EQUIVARIANT MIXED HODGE MODULES

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Abstract. These lecture notes were written to accompany the author's talk at the workshop on "Mixed Hodge Modules and their Applications," held at the Clay Mathematics Institute, Oxford, in August 2013.

1. Introduction

1.1. **Overview.** Let X be a smooth complex algebraic variety, and let G be an affine algebraic group acting on it. Each $g \in G$ determines an automorphism $a_g : X \to X$, and hence an autoequivalence a_g^* of the category of sheaves (or perverse sheaves, or $\mathscr{D}$-modules, or mixed Hodge modules) on X. Roughly, an *equivariant sheaf* is an object that is stable under all a_g^*. It is reasonably straightforward to make this into a precise definition. In the mixed Hodge module case, the resulting abelian category is denoted $\mathrm{MHM}_G(X)$. If G acts freely on X, it turns out that this is equivalent to $\mathrm{MHM}(X/G)$.

But we would also like to work at the derived level, and have available the formalism of (derived) sheaf functors. There are two obvious approaches that both go wrong:

- If we simply copy the definition of $\mathrm{MHM}_G(X)$ at the derived level, the resulting category is not a triangulated category.
- The naïve derived category $D^{\mathrm{b}}\mathrm{MHM}_G(X)$ does not, in general, admit well-behaved pull-back and push-forward functors.

So a more sophisticated approach is needed.

Here is a more precise statement of our desiderata. We want a rule that assigns to every smooth G-variety X a certain triangulated category $D^{\mathrm{b}}_G(X)$ with the following properties:

(1) $D^{\mathrm{b}}_G(X)$ should be equipped with a bounded t-structure whose heart is equivalent to the abelian category $\mathrm{MHM}_G(X)$.
(2) There should be a t-exact "forgetful functor" $\mathsf{For} : D^{\mathrm{b}}_G(X) \to D^{\mathrm{b}}\mathrm{MHM}(X)$.
(3) If $f : X \to Y$ is a smooth morphism of smooth G-varieties, say of relative dimension d, then there should be a t-exact triangulated functor $f^*[d] : D^{\mathrm{b}}_G(Y) \to D^{\mathrm{b}}_G(X)$ that "commutes with For" in the obvious sense.

Remarkably, it turns out that these three desiderata essentially force the definition of $D^{\mathrm{b}}_G(X)$ upon us! In fact, the third desideratum can be weakened to one involving only smooth acyclic maps. After the fact, we will see how define all the usual sheaf operations at the level of $D^{\mathrm{b}}_G(X)$ so that they commute with For. For now, we remark that if we restrict ourselves to varieties with a free G-action, then the "naïve" derived category $D^{\mathrm{b}}\mathrm{MHM}_G(X)$ (or $D^{\mathrm{b}}\mathrm{MHM}(X/G)$) *does* satisfy the desiderata.

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1.2. **Stacks on the acyclic site.** Before discussing the construction of $D^{\mathrm{b}}_G(X)$ in general, a word on the machinery of stacks is needed. Throughout this note, the word "stack" should be understood in the sense of [13, Chap. 19] or [10]. A stack is (roughly) just a sheaf of categories. (In particular, in this note, stacks are *not* assumed to be fibered in groupoids, and they are certainly not algebraic or Deligne–Mumford stacks.) The main point is that both objects and morphisms can be glued from local data in some (Grothendieck) topology. It is a shown in [5] that perverse sheaves form a stack in the étale and smooth topologies.

Unfortunately, derived categories of sheaves do *not* form a stack in most reasonable topologies. The difficulty is that push-forward from an open set need not be an exact functor, and that lack of exactness interferes with gluing objects from an open covering. (See [5, §3.2.1].)

In this note, we will circumvent this problem by defining a new Grothendieck topology on X, a very coarse topology called the *acyclic topology.* This topology is certainly useless for understanding the geometry of X; it is specifically designed to suppress geometric information, by enforcing some cohomology-vanishing on push-forward functors. But that cohomology-vanishing will imply that derived categories form a stack on the acyclic site.

1.3. **Definition of the equivariant derived category.** Here is the construction of $D^{\mathrm{b}}_G(X)$ in general.

(1) In the G-equivariant acyclic topology, any X admits a covering $\{u_i : U_i \to X\}_{i\in I}$ where G acts on each U_i freely.
(2) As noted earlier, we already know how to define $D^{\mathrm{b}}_G(U)$ when G acts freely on U: we simply put $D^{\mathrm{b}}_G(U) = D^{\mathrm{b}}\mathrm{MHM}_G(U)$. The assignment $U_i \mapsto D^{\mathrm{b}}\mathrm{MHM}_G(U_i)$ defines a prestack on $X_{G,\mathrm{acyc}}$.
(3) Let $\mathbf{D}_{G,X}$ be the stackification of that prestack, and define $D^{\mathrm{b}}_G(X)$ to be the cateogry of "global sections" of $\mathbf{D}_{G,X}$, i.e., its fiber over $X \xrightarrow{\mathrm{id}} X$.

Thus, an object of $D^{\mathrm{b}}_G(X)$ is, by definition, a compatible collection of objects in various $D^{\mathrm{b}}\mathrm{MHM}(U_i/G)$ as U_i ranges over a suitable covering family in the acyclic topology. In the case where X is a point and G is a torus, we will give a very concrete description of $D^{\mathrm{b}}_G(X)$.

1.4. **Comparison with Bernstein–Lunts.** The foundations of equivariant derived categories for constructible sheaves are developed in the monograph of Bernstein–Lunts [7]. (See also [12] for a self-contained treatment of the $\mathscr{D}$-module case.) The present note follows the ideas of [7] quite closely.

One minor difference in terminology between the present lecture and [7] concerns the notion of "acyclic maps." The precise definition will be given in Section 3; for now, we just note that an acyclic map $f : X \to Y$ ought to have two properties: (i) f_* should have vanishing (t-)cohomology in some range, and (ii) f^* should be (t-)exact. But with respect to what t-structure? In [7], Bernstein and Lunts work with the standard t-structure on the derived category of sheaves. Pullback along any map is t-exact for that t-structure, so condition (ii) imposes no restrictions. In this lecture, on the other hand, it makes more sense to impose (i) with respect to the natural t-structure on $D^{\mathrm{b}}\mathrm{MHM}(X)$, which corresponds (via rat) to the *perverse* t-structure on the derived category of sheaves. Since (ii) is no longer automatic, we also require acyclic maps to be smooth. Despite the difference in definition, many maps are acyclic in both our sense and that of [7]; see Exercise 3.4.

A more prominent difference with [7] is the central role given to the acyclic topology and stackification. In fact, the stackification perspective is implicit in [7]; indeed, they explicitly consider the assignment $U \mapsto D^{\mathrm{b}}_{\mathrm{c}}(U)$ as a fibered category. However, they focus exclusively on acyclic maps $U \to X$ where U has a *free* G-action, and such maps by themselves do not form a Grothendieck topology. Nevertheless, the discussion in [7, §2.4] amounts to describing $D^{\mathrm{b}}_{G}(X)$ by stackification.

Part 1. Mixed Hodge modules and Grothendieck topologies

2. Preliminaries

2.1. **Notation.** Given a smooth complex variety X, let X_{an} denote the corresponding analytic space. Let $\mathscr{O}_X$ denote the structure sheaf of X, and let $\mathscr{D}_X$ denote the sheaf of differential operators. Let $\mathrm{Mod}(\mathscr{D}_X)$ (resp. $\mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$, $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$) denote the category of all (resp. holonimic, regular holonomic) left $\mathscr{D}_X$-modules.

Fix a subfield $\mathbf{A} \subset \mathbb{R}$. Let $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{A})$, denote the derived category of algebraically constructible complexes of sheaves of $\mathbf{A}$-vector spaces on X_{an}, and let $\mathrm{Perv}(X, \mathbf{A}) \subset D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{A})$ denote the full subcategory of perverse $\mathbf{A}$-sheaves.

Let $\mathrm{MHM}(X, \mathbf{A})$ denote the abelian category of mixed Hodge $\mathbf{A}$-modules. This category comes equipped with the exact, faithful functor

$$\mathrm{rat} : \mathrm{MHM}(X, \mathbf{A}) \to \mathrm{Perv}(X, \mathbf{A}).$$

Recall [4] that the "realization functor" $D^{\mathrm{b}}\mathrm{Perv}(X, \mathbf{A}) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{A})$ is an equivalence of categories. We will identify $D^{\mathrm{b}}\mathrm{Perv}(X, \mathbf{A})$ with $D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{A})$, and regard the derived version of the functor above as a functor

$$\mathrm{rat} : D^{\mathrm{b}}\mathrm{MHM}(X, \mathbf{A}) \to D^{\mathrm{b}}_{\mathrm{c}}(X, \mathbf{A}).$$

In the sequel, we generally omit the field $\mathbf{A}$ from the notation for the categories defined above, instead writing $D^{\mathrm{b}}_{\mathrm{c}}(X)$, $\mathrm{Perv}(X)$, $\mathrm{MHM}(X)$, etc.

Let $\underline{\mathbf{A}}_X \in \mathrm{MHM}(X)$ denote the trivial mixed Hodge $\mathbf{A}$-module on X. Its underlying $\mathscr{D}_X$-module is $\mathscr{O}_X$, and $\mathrm{rat}(\underline{\mathbf{A}}_X)$ is (a shift of) the constant sheaf on X_{an} with value $\mathbf{A}$. The object $\underline{\mathbf{A}}_X$ is pure of weight $\dim X$.

Let $a_X : X \to \mathrm{pt}$ denote the constant map to a point. For $\mathcal{F}, \mathcal{G} \in \mathrm{MHM}(X)$, let

$$\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) = H^i(a_{X*}\, \mathrm{R}\mathscr{H}\!om(\mathcal{F}, \mathcal{G})).$$

Thus, $\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G})$ is an object of $\mathrm{MHM}(\mathrm{pt})$ equipped with a natural isomorphism

$$\mathrm{rat}\, \underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}^i_{D^{\mathrm{b}}_{\mathrm{c}}(X)}(\mathrm{rat}\, \mathcal{F}, \mathrm{rat}\, \mathcal{G}).$$

For an object $A \in D^{\mathrm{b}}\mathrm{MHM}(\mathrm{pt})$, we define its *Hodge cohomology* as follows:

$$H^i_{\mathrm{Hodge}}(A) := \mathrm{Hom}^i_{D^{\mathrm{b}}\mathrm{MHM}(\mathrm{pt})}(\underline{\mathbf{A}}_{\mathrm{pt}}, A).$$

When $A \in \mathrm{MHM}(\mathrm{pt})$, its Hodge cohomology $H^i_{\mathrm{Hodge}}(A)$ vanishes unless $i \in \{0, 1\}$.

For $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}\mathrm{MHM}(X)$, the main result of [15] implies that there is a natural short exact sequence

$$0 \to H^1_{\mathrm{Hodge}}(\underline{\mathrm{Hom}}^{i-1}(\mathcal{F}, \mathcal{G})) \to \mathrm{Hom}^i(\mathcal{F}, \mathcal{G}) \to H^0_{\mathrm{Hodge}}(\underline{\mathrm{Hom}}^i(\mathcal{F}, \mathcal{G})) \to 0. \tag{2.1}$$

2.2. **Smooth pullback.** Let $f : X \to Y$ be a smooth morphism of smooth complex varieties, say of relative dimension d. Then the functor $f^*[d] : D^{\mathrm{b}}\mathrm{MHM}(Y) \to D^{\mathrm{b}}\mathrm{MHM}(X)$ preserves the heart of the natural t-structure. (This follows from the corresponding statement for perverse sheaves; cf. [5, §4.2.4].) It will be convenient to introduce the notation

$$f^\dagger := f^*[d] : D^{\mathrm{b}}\mathrm{MHM}(Y) \to D^{\mathrm{b}}\mathrm{MHM}(X) \quad \text{or} \quad \mathrm{MHM}(Y) \to \mathrm{MHM}(X)$$

for this functor.

Proposition 2.1. *Let X and Y be smooth varieties, and let $f : X \to Y$ be a smooth surjective morphism with connected fibers. Then the natural maps*

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(f^\dagger \mathcal{F}, f^\dagger \mathcal{G}), \tag{2.2}$$

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(f^\dagger \mathcal{F}, f^\dagger \mathcal{G}) \tag{2.3}$$

are both isomorphisms.

Proof Sketch. The analogous statement in the setting of $D^{\mathrm{b}}_{\mathrm{c}}(X)$ appears in [5, Proposition 4.2.5]. The proof given there depends only on formal properties of the functors f_*, f^*, $\mathrm{R}\mathscr{H}om$, and a_{Y*}; the same argument also proves (2.2). Furthermore, since $f^\dagger$ is exact, we clearly have $\underline{\mathrm{Hom}}^{-1}(f^\dagger \mathcal{F}, f^\dagger \mathcal{G}) = 0$, and so by (2.1), $\mathrm{Hom}(f^\dagger \mathcal{F}, f^\dagger \mathcal{G}) \cong H^0_{\mathrm{Hodge}}(\underline{\mathrm{Hom}}(f^\dagger \mathcal{F}, f^\dagger \mathcal{G}))$. Thus, (2.2) implies (2.3). ☐

2.3. **The stack of mixed Hodge modules.** Let X_{sm} denote the small smooth site of X. Consider the prestack $\mathbf{M}$ on X_{sm} which associates to each smooth map $u : U \to X$ the category $\mathrm{MHM}(U)$, and which associates to a morphism

$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr, "u"'] & & V \arrow[dl, "v"] \\ & X & \end{tikzcd}$$

the functor $i^\dagger : \mathrm{MHM}(V) \to \mathrm{MHM}(U)$. In this subsection, we show that this prestack is a stack.

Proposition 2.2. *Let $\mathcal{F}, \mathcal{G} \in \mathrm{MHM}(X)$, and let $\mathbf{Hom}(\mathcal{F}, \mathcal{G})$ denote the presehaf on X_{sm} given by*

$$\mathbf{Hom}(\mathcal{F}, \mathcal{G})(U \xrightarrow{u} X) = \mathrm{Hom}(u^\dagger \mathcal{F}, u^\dagger \mathcal{G}).$$

This presheaf is a sheaf. In other words, the prestack $\mathbf{M}$ is separated.

Proof. We begin with various analogues of $\mathbf{Hom}(\mathcal{F}, \mathcal{G})$. Let

$$\mathbf{Hom}_{\mathscr{D}}(\mathcal{F}, \mathcal{G}), \quad \mathbf{Hom}_{\mathscr{D}F}(\mathcal{F}, \mathcal{G}), \quad \mathbf{Hom}_{\mathrm{Perv}}(\mathcal{F}, \mathcal{G}), \quad \mathbf{Hom}_{\mathrm{Perv},W}(\mathcal{F}, \mathcal{G})$$

be the presheaves that assign to $u : U \to X$ the space of morphisms between the underlying $\mathscr{D}$-modules (resp. filtered $\mathscr{D}$-modules (with the Hodge filtration); perverse sheaves; filtered perverse sheaves (with the weight filtration)) of $u^\dagger \mathcal{F}$ and $u^\dagger \mathcal{G}$. Now, $\mathbf{Hom}_{\mathscr{D}}(\mathcal{F}, \mathcal{G})$ is just the usual "internal $\mathscr{H}om$" for $\mathscr{D}$-modules; it is an elementary observation that this presheaf is a sheaf. The same elementary considerations show that $\mathbf{Hom}_{\mathscr{D}F}(\mathcal{F}, \mathcal{G})$ is also a sheaf.

Next, according to [5, Corollaire 2.1.22], $\mathbf{Hom}_{\mathrm{Perv}}(\mathcal{F}, \mathcal{G})$ is a sheaf. (To be precise, that result treats the analytic site. The analogous result for the étale site is discussed in [5, §2.1.24 and §2.2.19], and similar remarks apply to the smooth site.) It is straightforward to see that the subpresheaf $\mathbf{Hom}_{\mathrm{Perv},W}(\mathcal{F},\mathcal{G}) \subset \mathbf{Hom}_{\mathrm{Perv}}(\mathcal{F},\mathcal{G})$ is also a sheaf.

Now, both $\mathbf{Hom}_{\mathscr{D}F}(\mathcal{F},\mathcal{G})$ and $\mathbf{Hom}_{\mathrm{Perv},W}(\mathcal{F},\mathcal{G})$ can be identified with subsheaves of $\mathbf{Hom}_{\mathscr{D}}(\mathcal{F},\mathcal{G})$ (for the latter, this involves extending scalars to $\mathbb{C}$ and the Riemann–Hilbert correspondence). Under these identifications, the presheaf $\mathbf{Hom}(\mathcal{F},\mathcal{G})$ is simply the intersection $\mathbf{Hom}_{\mathscr{D}F}(\mathcal{F},\mathcal{G}) \cap \mathbf{Hom}_{\mathrm{Perv},W}(\mathcal{F},\mathcal{G})$, and such an intersection is automatically a sheaf. □

Theorem 2.3. *The prestack* $\mathbf{M} : U \mapsto \mathrm{MHM}(U)$ *on* X_{sm} *is a stack.*

The corresponding statement for $\mathscr{D}$-modules is elementary (see, for instance, [13, Proposition 19.4.7(b)]). For perverse sheaves, this is proved in [5, §2.2.19]. However, in contrast with Proposition 2.2, we cannot simply deduce the result for mixed Hodge modules from those for $\mathscr{D}$-modules and perverse sheaves—that would just yield the weaker statement that the categories $\mathrm{MF}_{rh}\mathrm{W}(U)$ (for $U \xrightarrow{u} X$ smooth) form a stack. (For the definition of $\mathrm{MF}_{rh}\mathrm{W}(U)$, see the discussion following [14, Theorem 0.1].) Instead, we essentially copy the proof of [5, §2.2.19].

Proof. Proposition 2.2 already shows that one can glue morphisms of mixed Hodge modules. As explained in [5, §2.1.23 and §2.2.19] or [13, Proposition 19.4.6(v)], it remains to show that one can glue objects. Let $\{u_i : U_i \to X\}_{i\in I}$ be a covering of X in the smooth topology. For brevity, let $U_{ij} = U_i \times_X U_j$ and $U_{ijk} = U_i \times_X U_j \times_X U_k$. The natural projection maps are denoted

$$\begin{array}{ll} r^1_{ij} : U_{ij} \to U_i, & r^{12}_{ijk} : U_{ijk} \to U_{ij}, \\ & r^{13}_{ijk} : U_{ijk} \to U_{ik}, \\ r^2_{ij} : U_{ij} \to U_j, & r^{23}_{ijk} : U_{ijk} \to U_{jk}, \end{array} \tag{2.4}$$

and suppose we are given the following data:

1. for each $i \in I$, an object $\mathcal{F}_i \in \mathrm{MHM}(U_i)$.
2. for each pair $(i,j) \in I \times I$, an isomorphism $\varphi_{ij} : r^{1*}_{ij}\mathcal{F}_i \xrightarrow{\sim} r^{2*}_{ij}\mathcal{F}_j$.

These data are required to satisfy the property that the following diagram commutes (cf. [13, §19.3], and especially [13, Eq. (19.3.10)]):

$$\begin{tikzcd} r^{12*}_{ijk} r^{1*}_{ij}\mathcal{F}_i \arrow[r, "r^{12*}_{ijk}\varphi_{ij}"] \arrow[d, "\wr"'] & r^{12*}_{ijk} r^{2*}_{ij}\mathcal{F}_j \arrow[r, "\sim"] & r^{23*}_{ijk} r^{1*}_{jk}\mathcal{F}_j \arrow[d, "r^{23*}_{ijk}\varphi_{jk}"] \\ r^{13*}_{ijk} r^{1*}_{ik}\mathcal{F}_i \arrow[r, "r^{13*}_{ijk}\varphi_{ik}"'] & r^{13*}_{ijk} r^{2*}_{ik}\mathcal{F}_k \arrow[r, "\sim"'] & r^{23*}_{ijk} r^{2*}_{jk}\mathcal{F}_k \end{tikzcd} \tag{2.5}$$

We must show that there exists an object $\mathcal{F} \in \mathrm{MHM}(X)$ together with isomorphisms $\varphi_i : u_i^\dagger \mathcal{F} \xrightarrow{\sim} \mathcal{F}_i$ such that

$$\begin{tikzcd} r^{1\dagger}_{ij} u_i^\dagger \mathcal{F} \arrow[r, "\sim"] \arrow[d, "r^{1\dagger}_{ij}\varphi_i"'] & r^{2\dagger}_{ij} u_j^\dagger \mathcal{F} \arrow[d, "r^{2\dagger}_{ij}\varphi_j"] \\ r^{1\dagger}_{ij}\mathcal{F}_i \arrow[r, "\varphi_{ij}"'] & r^{2\dagger}_{ij}\mathcal{F}_j \end{tikzcd} \tag{2.6}$$

commutes. As mentioned above, the construction is copied from [5, §2.2.19]. For each $(i,j) \in I \times I$, there are adjunction maps $\eta^1_{ij} : \mathcal{F}_i \to r^1_{ij\dagger} r^{1\dagger}_{ij}\mathcal{F}_i$ and $\eta^2_{ij} : \mathcal{F}_j \to$

$r^2_{ij\dagger} r^{2\dagger}_{ij} \mathcal{F}_j$. Let

$$\mathcal{U} = \prod_{i \in I} u_{i\dagger} \mathcal{F}_i \qquad \text{and} \qquad \mathcal{V} = \prod_{i,j \in I} u_{i\dagger} r^1_{ij\dagger} r^{1\dagger}_{ij} \mathcal{F}_i \cong \prod_{i,j \in I} u_{j\dagger} r^2_{ij\dagger} r^{2\dagger}_{ij} \mathcal{F}_j.$$

Form the maps $\bar{\eta}^1 = (u_{i\dagger}\eta^1_{ij})_{i,j\in I} : \mathcal{U} \to \mathcal{V}$ and $\bar{\eta}^2 = (u_{j\dagger}\eta^2_{ij})_{i,j\in I} : \mathcal{U} \to \mathcal{V}$, and let $f = \bar{\eta}^1 - \bar{\eta}^2$. Finally, let $\mathcal{K}$ be the cocone of f, so that we have a distinguished triangle

$$\mathcal{K} \to \mathcal{U} \xrightarrow{f} \mathcal{V} \to$$

in $D^{\mathrm{b}}\mathrm{MHM}(X)$. The exactness properties of the various sheaf operations used here imply that $H^i(\mathcal{U})$ and $H^i(\mathcal{V})$ vanish for $i < 0$, so the same holds for $H^i(\mathcal{K})$. In particular, the sequence

$$0 \to H^0(\mathcal{K}) \to H^0(\mathcal{U}) \to H^0(\mathcal{V})$$

is exact. Let $\mathcal{F} = H^0(\mathcal{K})$. The required isomorphisms φ_i come from the adjunction $(u_i^\dagger, u_{i\dagger})$. The details are explained in [5, §2.2.19]. $\square$

This result has the following consequence, which is not obvious, at least to me.

Corollary 2.4. *For an object of* $\mathrm{MF}_{rh}\mathrm{W}(X)$*, the property of being a mixed Hodge module is local in the smooth topology.*

3. Acyclic maps

For any $n \in \mathbb{Z}$, we have the usual truncation functors

$$\tau_{\le n}, \tau_{\ge n} : D^{\mathrm{b}}\mathrm{MHM}(X) \to D^{\mathrm{b}}\mathrm{MHM}(X).$$

The same notation will also be used for truncation with respect to the natural t-structure on $D^{\mathrm{b}}\mathrm{Perv}(X)$, or the perverse t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X)$. (The natural t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X)$ will not appear in this lecture, except in Exercise 3.4.) We write H^i for cohomology with respect to the natural t-structure on $D^{\mathrm{b}}\mathrm{MHM}(X)$ or the perverse t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X)$. Given integers $a \le b$, let $D^{[a,b]}\mathrm{MHM}(X)$ denote the full subcategory of $D^{\mathrm{b}}\mathrm{MHM}(X)$ consisting of objects $\mathcal{F}$ with $H^i(\mathcal{F}) = 0$ unless $a \le i \le b$.

Definition 3.1. A smooth morphism of smooth complex varieties $f : X \to Y$ is said to be *n-acyclic* (resp. *∞-acyclic*) if for every smooth morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ has the property that for all $\mathcal{F} \in \mathrm{MHM}(Y')$, the natural morphism

$$\mathcal{F} \to \tau_{\le n} f'_\dagger f'^\dagger \mathcal{F} \qquad \text{resp.} \qquad \mathcal{F} \to f'_\dagger f'^\dagger \mathcal{F}$$

is an isomorphism. A 0-acyclic map is simply said to be *acyclic.*

Of course, if $n \le m$, then every m-acyclic map is also n-acyclic. The following lemma is immediate from the definition.

Lemma 3.2. *Let $f : X \to Y$ be an acyclic map. Then f is surjective, and for $\mathcal{F} \in \mathrm{MHM}(Y)$, we have $\mathcal{F} = 0$ if and only if $f^\dagger \mathcal{F} = 0$.* $\square$

Exercise 3.3. Let X be a smooth, connected variety, and assume that $H^i(X) = 0$ for $i = 1, \ldots, n$. Then the constant map $X \to \mathrm{pt}$ is n-acyclic.

Exercise 3.4. As noted in the introduction, the definition above differs from that in [7]. To distinguish the two, we rename the notion from [7] "n-standard-acyclic." Let us write ${}^{\mathrm{std}}\tau_{\leq n}$ for truncation with respect to the standard t-structure on $D^{\mathrm{b}}_{\mathrm{c}}(X)$. Recall that a map $f : X \to Y$ is n-standard acyclic if for every base change $f' : X \times_Y Y' \to Y'$ and every constructible sheaf $\mathcal{F}$ on Y', the morphism $\mathcal{F} \to {}^{\mathrm{std}}\tau_{\leq n} f'_* f'^* \mathcal{F}$ is an isomorphism. (Note that f is *not* assumed to be smooth.)

Prove that if $f : X \to Y$ is $(n + \dim Y)$-standard-acyclic, then it is n-acyclic. Is there an implication in the other direction?

Lemma 3.5. *Suppose $f : X \to Y$ is n-acyclic, and let $a \leq b$ be integers with $b - a < n$. For all $\mathcal{F} \in D^{[a,b]}\mathrm{MHM}(X)$, the morphism $\mathcal{F} \to \tau_{\leq b} f_\dagger f^\dagger \mathcal{F}$ is an isomorphism.*

Proof. We proceed by induction on $b - a$. If $b = a$, the result is immediate from the definition. Otherwise, let $\mathcal{F}' = \tau_{\leq a}\mathcal{F}$ and $\mathcal{F}'' = \tau_{\geq a+1}\mathcal{F}$, and consider the distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$. Applying $f_\dagger f^\dagger$ to this and then forming the associated long exact sequences in cohomology, we obtain the commutative diagram

$$\begin{tikzcd}
H^{i-1}(\mathcal{F}'') \arrow[r] \arrow[d] & H^i(\mathcal{F}') \arrow[r] \arrow[d] & H^i(\mathcal{F}) \arrow[r] \arrow[d] & H^i(\mathcal{F}'') \arrow[r] \arrow[d] & H^{i+1}(\mathcal{F}') \arrow[d] \\
H^{i-1}(f_\dagger f^\dagger \mathcal{F}'') \arrow[r] & H^i(f_\dagger f^\dagger \mathcal{F}') \arrow[r] & H^i(f_\dagger f^\dagger \mathcal{F}) \arrow[r] & H^i(f_\dagger f^\dagger \mathcal{F}'') \arrow[r] & H^{i+1}(f_\dagger f^\dagger \mathcal{F}')
\end{tikzcd}$$

By induction, the first, second, and fourth vertical arrows are isomorphisms whenever $a \leq i \leq b$. The fifth vertical arrow is an isomorphism when $a \leq i \leq b-1$, and it is obviously injective when $i = b$. By the five lemma, $H^i(\mathcal{F}) \to H^i(f_\dagger f^\dagger \mathcal{F})$ is an isomorphism for $a \leq i \leq b$, so $\mathcal{F} \to \tau_{\leq b} f_\dagger f^\dagger \mathcal{F}$ is an isomorphism. □

Corollary 3.6. *Suppose $f : X \to Y$ is n-acyclic, and let $a \leq b$ be integers with $b - a < n$. Then $f^\dagger : D^{[a,b]}\mathrm{MHM}(Y) \to D^{[a,b]}\mathrm{MHM}(X)$ is fully faithful.*

Proof. For $\mathcal{F}, \mathcal{G} \in D^{[a,b]}\mathrm{MHM}(Y)$, we have $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, \tau_{\leq b} f_\dagger f^\dagger \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_\dagger f^\dagger \mathcal{G}) \cong \mathrm{Hom}(f^\dagger \mathcal{F}, f^\dagger \mathcal{G})$. □

Lemma 3.7. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth complex varieties.*

1. *If f and g are both n-acyclic, then $g \circ f$ is n-acyclic.*
2. *If $g \circ f$ is n-acyclic and g is acyclic, then g is in fact n-acyclic.*

Proof. (1) Let $\mathcal{F} \in \mathrm{MHM}(Z)$, and consider the object $g^\dagger \mathcal{F} \in \mathrm{MHM}(Y)$. Since f is n-acyclic, we have $g^\dagger \mathcal{F} \cong \tau_{\leq n} f_\dagger f^\dagger g^\dagger \mathcal{F}$. In other words, there is a distinguished triangle $g^\dagger \mathcal{F} \to f_\dagger f^\dagger g^\dagger \mathcal{F} \to \tau_{>n} f_\dagger f^\dagger g^\dagger \mathcal{F} \to$, and hence also

$$g_\dagger g^\dagger \mathcal{F} \to g_\dagger f_\dagger f^\dagger g^\dagger \mathcal{G} \to g_\dagger \tau_{>n} f_\dagger f^\dagger g^\dagger \mathcal{F} \to .$$

Since $g_\dagger$ is left t-exact, we have that $H^i(g_\dagger \tau_{>n} f_\dagger f^\dagger g^\dagger \mathcal{F}) = 0$ for $i \leq n$. Therefore, we have isomorphisms $H^i(g_\dagger g^\dagger \mathcal{F}) \xrightarrow{\sim} H^i(g_\dagger f_\dagger f^\dagger g^\dagger \mathcal{F})$ for $i \leq n$. In other words, the natural map $\tau_{\leq n} g_\dagger g^\dagger \mathcal{F} \to \tau_{\leq n} g_\dagger f_\dagger f^\dagger g^\dagger \mathcal{F}$ is an isomorphism. Since $\mathcal{F} \to \tau_{\leq n} g_\dagger g^\dagger \mathcal{F}$ is also an isomorphism, the composition $\mathcal{F} \to \tau_{\leq n}(g \circ f)_\dagger (g \circ f)^\dagger \mathcal{F}$ is as well, as desired.

(2) Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[rr] \arrow[d, "\wr"'] & & \tau_{\leq n} g_\dagger g^\dagger \mathcal{F} \arrow[dll] \arrow[d, "\wr"] \\
\tau_{\leq n} g_\dagger f_\dagger f^\dagger g^\dagger \mathcal{F} \arrow[rr] & & \tau_{\leq n} g_\dagger f_\dagger f^\dagger g^\dagger \tau_{\leq n} g_\dagger g^\dagger \mathcal{F}
\end{tikzcd}$$

The vertical arrows are isomorphisms because $g \circ f$ is assumed to be n-acyclic. (For the right-hand vertical arrow, use Lemma 3.5. This diagram shows that the diagonal arrow $\tau_{\leq n} g_\dagger g^\dagger \mathcal{F} \to \tau_{\leq n} g_\dagger f_\dagger f^\dagger g^\dagger \mathcal{F}$ has both left and right inverses, so it is an isomorphism. We then conclude that the top horizontal arrow is an isomorphism as well, as desired. □

Definition 3.8. Let $\mathrm{Acyc}(X)$ be the category whose objects are acyclic maps $U \to X$, and in which the morphisms $(U \xrightarrow{u} X) \to (V \xrightarrow{v} X)$ are the acyclic maps $f : U \to V$ such that $v \circ f = u$. Such a morphism in $\mathrm{Acyc}(X)$ is said to be *n-acyclic* if the underlying morphism of smooth varieties is n-acyclic in the sense of Definition 3.1.

Let $\mathcal{V}$ be an object in $\mathrm{Acyc}(X)$, and let $\{f_i : \mathcal{U}_i \to \mathcal{V}\}_{i \in I}$ be a family of morphisms. This family is said to be a *covering family* if for every integer $n \geq 0$, there exists an $i \in I$ such that $f_i : \mathcal{U}_i \to \mathcal{V}$ is n-acyclic. This defines a Grothendieck topology on $\mathrm{Acyc}(X)$, called the *acyclic topology*.

The *acyclic site* of X, denoted X_{acyc}, is the site consisting of $\mathrm{Acyc}(X)$ together with the acyclic topology.

Exercise 3.9. Check that this definition satisfies the axioms for a Grothendieck topology. (*Hint*: Lemma 3.7(2) plays a key role at several points in the proof.)

The acyclic topology is useless for doing actual topology, as the following exercise shows. But it will be useful to us as a way to organize certain desirable properties of $D^{\mathrm{b}}\mathrm{MHM}(X)$, and to impose desired properties on the equivariant derived category.

Exercise 3.10. Let $H^\bullet_{\mathrm{acyc}}(X, \mathbf{A})$ denote the "acyclic cohomology" of X, i.e., the sheaf cohomology in the acyclic topology of the constant sheaf with value $\mathbf{A}$. Prove that $H^0_{\mathrm{acyc}}(X, \mathbf{A}) \cong \mathbf{A}$, and that $H^i_{\mathrm{acyc}}(X, \mathbf{A}) = 0$ for all $i > 0$. (*Hint*: See [13, §18.7] for a help converting this to a Čech cohomology question. I admit that I have not worked through the details myself!)

Let $\mathbf{D}_X$ be the prestack on X_{acyc} given by $\mathbf{D}_X(U \xrightarrow{u} X) = D^{\mathrm{b}}\mathrm{MHM}(U)$. The next two results are analogues of those in Section 2.3.

Proposition 3.11. *Let $\mathcal{F}, \mathcal{G} \in D^{\mathrm{b}}\mathrm{MHM}(X)$, and let $\mathbf{Hom}(\mathcal{F}, \mathcal{G})$ denote the presehaf on X_{acyc} given by*

$$\mathbf{Hom}(\mathcal{F}, \mathcal{G})(U \xrightarrow{u} X) = \mathrm{Hom}(u^\dagger \mathcal{F}, u^\dagger \mathcal{G}).$$

This presheaf is a sheaf. In other words, the prestack $\mathbf{D}_X$ is separated.

Proof Sketch. Choose integers $a \leq b$ such that $\mathcal{F}$ and $\mathcal{G}$ both lie in $D^{[a,b]}\mathrm{MHM}(X)$, and let $n > b - a$. Consider a covering family $\{u_i : U_i \to X\}_{i,I}$ in X_{acyc}. For any $u_i : U_i \to X$ that is at least n-acyclic, Corollary 3.6 tells us that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(u_i^\dagger \mathcal{F}, u_i^\dagger \mathcal{G})$ is an isomorphism. It is easily deduced from this that $\mathbf{Hom}(\mathcal{F}, \mathcal{G})$ is a sheaf. □

Theorem 3.12. *The prestack* $\mathbf{D}_X : U \mapsto D^{\mathrm{b}}\mathrm{MHM}(U)$ *on* X_{acyc} *is a stack.*

Proof. As in the proof of Theorem 2.3, it suffices to explain how to glue objects. The task here is rather easier, simply because every acyclic map is surjective.

Let $\{u_i : U_i \to X\}_{i\in I}$ be a covering of X in the acyclic topology. We will use the notation of (2.4) and the subsequent text. In particular, we have objects $\mathcal{F}_i \in D^{\mathrm{b}}\mathrm{MHM}(U_i)$ and isomorphisms $\varphi_{ij} : r^{1\dagger}_{ij}\mathcal{F}_i \xrightarrow{\sim} r^{2\dagger}_{ij}\mathcal{F}_j$ satisfying (2.5).

Note that for any $k \in \mathbb{Z}$ and any $i, j \in I$, we have that $H^k(\mathcal{F}_i) = 0$ if and only if $H^k(r^{1\dagger}_{ij}\mathcal{F}_i) = 0$, by Lemma 3.2. But $H^k(r^{1\dagger}_{ij}\mathcal{F}_i) \cong H^k(r^{2\dagger}_{ij}\mathcal{F}_j)$ vanishes if and only if $H^k(\mathcal{F}_j) = 0$. In other words, for each $k \in \mathbb{Z}$, the object $H^k(\mathcal{F}_i)$ must either vanish for all i or be nonzero for all i. Thus, we can choose an interval $[a, b] \subset \mathbb{Z}$ such that $\mathcal{F}_i \in D^{[a,b]}\mathrm{MHM}(U_i)$ for some, and hence all, $i \in I$.

Now, let $n = b - a + 1$, and choose some $i_0 \in I$ such that $u_{i_0} : U_{i_0} \to X$ is n-acyclic. Let $\mathcal{F} = \tau_{\le b} u_{i_0\dagger}\mathcal{F}_{i_0}$. For each $j \in I$, define $\varphi_j : u^\dagger_j\mathcal{F} \xrightarrow{\sim} \mathcal{F}_j$ to be the composition of the maps

$$u^\dagger_j\mathcal{F} \xrightarrow{\text{base change}} \tau_{\le b} r^2_{i_0j\dagger} r^{1\dagger}_{i_0j}\mathcal{F}_{i_0} \xrightarrow{\tau_{\le b}\varphi_{i_0j}} \tau_{\le b} r^2_{i_0j\dagger} r^{2\dagger}_{i_0j}\mathcal{F}_j \xrightarrow{\text{Lemma 3.5}} \mathcal{F}_j.$$

Each of the three maps here is an isomorphism. The first comes from the following cartesian diagram:

$$\begin{tikzcd} U_{i_0j} \arrow[r, "r^1_{i_0j}"] \arrow[d, "r^2_{i_0j}"'] & U_{i_0} \arrow[d, "u_{i_0}"] \\ U_j \arrow[r, "u_j"'] & X \end{tikzcd}$$

It is left to the reader to check that (2.6) holds. $\square$

Part 2. Equivariant categories of mixed Hodge modules

4. Digression: Sheaves of sets

This short and very elementary section is based on my experiences trying to convince graduate students that the usual definition of "equivariant sheaf" is the correct one—in other words, that that definition does indeed encode the intuitive notion that it ought to. This section is independent of the rest of the lecture; other than some notation, it will not be referred to again. Readers who are happy with the definition of equivariant sheaves should skip it.

In this section, X will be denote a topological space, and G will denote a topological group acting on it. Let

$$a : G \times X \to X \qquad \text{and} \qquad p : G \times X \to X \tag{4.1}$$

denote the action map and the projection onto the second factor, respectively. We will also make use the following maps:

$$\begin{aligned} s &: X \to G \times X & s(x) &= (e, x), \\ m &: G \times G \times X \to G \times X & m(g, h, x) &= (gh, x), \\ b &: G \times G \times X \to G \times X & m(g, h, x) &= (g, a(h, x)) \\ q &: G \times G \times X \to G \times X & q(g, h, x) &= (h, x) \end{aligned} \tag{4.2}$$

These maps satisfy the following relations, which encode the fact that a is an action:

$$a \circ s = \mathrm{id} = p \circ s, \qquad a \circ m = a \circ b, \qquad p \circ q = p \circ m, \qquad a \circ q = p \circ b$$

Definition 4.1. A *G-equivariant sheaf of sets on X* is a pair (A, β), where A is a sheaf of sets on X, and $\beta : p^*A \xrightarrow{\sim} a^*A$ is an isomorphism satisfying the following two axioms:

(1) $s^*\beta = \mathrm{id}$.
(2) $b^*\beta \circ q^*\beta = m^*\beta$.

To make sense of this definition, we will switch to the language of étalé spaces. Recall that an étalé space over X is simply a local homeomorphism $\pi : E \to X$. There is an equivalence of categories between étalé spaces over X and sheaves of sets on X.

Definition 4.2. A *G-equivariant étalé space on X* is an étalé space $\pi : E \to X$ over X together with a G-action $a_E : G \times E \to E$ such that π is G-equivariant.

Note that $\mathrm{id} \times \pi : G \times E \to G \times X$ is an étalé space over $G \times X$. Indeed, it is the pullback p^*E. We can also describe a^*E: it is the étalé space $\rho : G \times E \to G \times X$ with

$$\rho(g, y) = (g, g^{-1} \cdot \pi(y)).$$

Given a G-equivariant étalé space over X, we can define a map $\hat\beta : G \times E \to G \times E$ by $\hat\beta(g, x) = (g, g \cdot x)$. This map makes the following diagram commute:

$$\begin{tikzcd}
G \times E \arrow[rr, "\hat\beta"] \arrow[dr, "\mathrm{id}\times\pi"'] & & G \times E \arrow[dl, "\rho"] \\
& G \times X &
\end{tikzcd}$$

That is, $\hat\beta$ is an isomorphism of étalé spaces over $G \times X$. Conversely, given $\hat\beta$, we can recover $a_E = p \circ \hat\beta$.

Exercise 4.3. Let $\hat\beta : a^*E \to p^*E$ be an isomorphism of étalé spaces over $G \times X$. Prove that the following conditions are equivalent:

(1) The map $p \circ \hat\beta : G \times E \to E$ defines an action of G on E. Moreover, this action makes E into a G-equivariant étalé space over X.
(2) The map $\hat\beta$ has the following two properties:
   (a) $s^*\hat\beta = \mathrm{id} : E \to E$.
   (b) $b^*\hat\beta \circ q^*\hat\beta = m^*\hat\beta$.

Then, prove that there is an equivalence of categories between G-equivariant étalé spaces over X and G-equivariant sheaves of sets on X.

5. The abelian category $\mathrm{MHM}_G(X)$

From now on, X will denote a smooth complex algebraic variety, and G will denote an affine algebraic group acting on X. Let a, p, s, m, b, and q be as in (4.1) and (4.2).

Definition 5.1. A *G-equivariant mixed Hodge module on X* is a pair $(\mathcal{F}, \beta)$, where $\mathcal{F} \in \mathrm{MHM}(X)$, and $\beta : p^*\mathcal{F} \xrightarrow{\sim} a^*\mathcal{F}$ is an isomorphism satisfying the following two axioms:

(1) $s^*\beta = \mathrm{id}$.
(2) $b^*\beta \circ q^*\beta = m^*\beta$.

A *morphism of G-equivariant mixed Hodge modules* $f : (\mathcal{F}, \beta) \to (\mathcal{G}, \gamma)$ is a morphism $f : A \to A'$ in $\mathrm{MHM}(X)$ such that the following diagram commutes:

$$\begin{tikzcd} p^*\mathcal{F} \arrow[r, "\beta"] \arrow[d, "p^*f"'] & a^*\mathcal{F} \arrow[d, "a^*f"] \\ p^*\mathcal{G} \arrow[r, "\gamma"'] & a^*\mathcal{G} \end{tikzcd} \tag{5.1}$$

Let $\mathrm{MHM}_G(X)$ denote the category of G-equivariant mixed Hodge modules on X. It is easy to see this this is naturally an abelian category. We have an obvious forgetful functor

$$\mathsf{For} : \mathrm{MHM}_G(X) \to \mathrm{MHM}(X).$$

A priori, a G-equivariant mixed Hodge module is a mixed Hodge module together with additional data. However, the next theorem tells us that when G is connected, there can be at most one choice for that additional data. In other words, when G is connected, equivariance is a property, rather than additional data.

Theorem 5.2. *If G is connected, then the forgetful functor* $\mathsf{For} : \mathrm{MHM}_G(X) \to \mathrm{MHM}(X)$ *is fully faithful.*

Proof. Let $\mathcal{F}, \mathcal{G} \in \mathrm{MHM}_G(X)$. For brevity, we will use the same notation to denote the corresponding objects in $\mathrm{MHM}(X)$, rather than writing $\mathsf{For}(\mathcal{F})$ and $\mathsf{For}(\mathcal{G})$. We write $\mathrm{Hom}_G(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ for the morphism spaces in $\mathrm{MHM}_G(X)$ and in $\mathrm{MHM}(X)$, respectively. Consider the following diagram:

$$\begin{tikzcd} & \mathrm{Hom}(a^*\mathcal{F}, a^*\mathcal{G}) \arrow[r, "s^*"] \arrow[dd, "\gamma^{-1}\circ(-)\circ\beta"] & \mathrm{Hom}(s^*a^*\mathcal{F}, s^*a^*\mathcal{G}) \arrow[dd, equal] \\ \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \arrow[ur, "a^*", "\sim"'] \arrow[dr, "p^*"', "\sim"] \arrow[urr, equal, bend left=40] \arrow[drr, equal, bend right=40] \arrow[r, phantom, "?"] & & \\ & \mathrm{Hom}(p^*\mathcal{F}, p^*\mathcal{G}) \arrow[r, "s^*"'] & \mathrm{Hom}(s^*p^*\mathcal{F}, s^*p^*\mathcal{G}) \end{tikzcd} \tag{5.2}$$

The upper triangle (involving a^*, s^*, and an equality) commutes, as does the bottom triangle. The square on the right-hand side of the diagram also commutes. Only the triangle marked ? is not known a priori to commute. But Proposition 2.1 implies that the maps a^* and p^* are isomorphisms, so the triangle ? must commute as well.

By definition, $\mathrm{Hom}_G(\mathcal{F}, \mathcal{G})$ is the subset of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ for which (5.1) commutes. But the commutativity of ? above implies that every element of $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ has that property, so $\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. □

6. Free G-spaces

Definition 6.1. The action of G on X is said to be *free* if it admits a geometric quotient $\phi : X \to X/G$ that is locally trivial in the étale topology.

For the notion of *geometric quotient*, see [11, Definition 0.6]. Note that the definition of *free* in [11, Definition 0.8] is different (and weaker than this one): there, an action is free if the map $\Psi := (a, p) : G \times X \to X \times X$ is a closed immersion. As explained in [9, §3], the condition in Definition 6.1 above implies the one in [11, Definition 0.8], but the converse fails in general. The stronger condition in Definition 6.1 is better suited to our purposes (cf. [7, §0.3]).

Theorem 6.2. *Assume that G acts freely on X, and let $\phi : X \to X/G$ be the quotient map. Then $\phi^\dagger$ induces an equivalence of categories*

$$\phi^\dagger : \mathrm{MHM}(X/G) \xrightarrow{\sim} \mathrm{MHM}_G(X).$$

Proof Sketch. This result essentially follows from the results of §2.3 and the observation that the smooth map $\{\phi : X \to X/G\}$ constitutes a covering of X/G in the smooth topology. To spell this out in a bit more detail, consider the commutative diagram

$$\begin{tikzcd} G \times X \arrow[r, "a"] \arrow[d, "p"'] & X \arrow[d, "\phi"] \\ X \arrow[r, "\phi"'] & X/G \end{tikzcd}$$

It is easy to see that this square is cartesian, so we can identify $G \times X \cong X \times_{X/G} X$. Likewise, the diagram below gives us an identification $G \times G \times X \cong X \times_{X/G} \times_{X/G} X$.

$$\begin{tikzcd} G \times G \times X \arrow[rr, "b"] \arrow[dr, "m"] \arrow[dd, "q"'] & & G \times X \arrow[dr, "a"] \arrow[dd, "p"] & \\ & G \times X \arrow[rr, "a"] \arrow[dd, "p"] & & X \arrow[dd, "\phi"] \\ G \times X \arrow[rr, "a"] \arrow[dr, "p"'] & & X \arrow[dr, "\phi"] & \\ & X \arrow[rr, "\phi"'] & & X/G \end{tikzcd}$$

Under these identifications, the various projection maps in (2.4) correspond to the maps in (4.1)–(4.2) as follows:

$$\begin{aligned} r^1 &= p, \\ r^2 &= a, \end{aligned} \qquad \begin{aligned} r^{12} &= q, \\ r^{13} &= m, \\ r^{23} &= b. \end{aligned}$$

Condition (2) of Definition 5.1 is just a restatement of the condition that the diagram in (2.5) commutes.

With this perspective, it is clear that for any $\mathcal{F} \in \mathrm{MHM}(X/G)$, the object $\phi^\dagger\mathcal{F} \in \mathrm{MHM}(X)$ comes equipped with a natural isomorphism $\beta : p^*\phi^\dagger\mathcal{F} \xrightarrow{\sim} a^*\phi^\dagger\mathcal{F}$ satisfying condition (2) of Definition 5.1. Condition (1) also holds: it is a consequence of the fact that $p \circ s = a \circ s = \mathrm{id}$, combined with general results about pullback functors and compositions, explained, for instance, in [2, Lemmas B.4(c) and B.6(c)]. Thus, it makes sense to regard $\phi^\dagger$ as taking values in $\mathrm{MHM}_G(X)$ rather than in $\mathrm{MHM}(X)$.

Now, Proposition 2.2 tells us that specifying a morphism in $\mathrm{MHM}(X/G)$ is equivalent to specifying it locally with respect to some covering in the smooth topology.

For the covering $\{\phi : X \to X/G\}$, "specifically a morphism locally" just means giving a morphism in $\mathrm{MHM}(X)$ that satisfies (5.1). In other words, Proposition 2.2 implies that $\phi^\dagger : \mathrm{MHM}(X/G) \to \mathrm{MHM}_G(X)$ is fully faithful. Similar considerations with Theorem 2.3 show that $\phi^\dagger$ is also essentially surjective, so it is an equivalence of categories. $\square$

Exercise 6.3. Suppose G acts on X, and suppose that $H \trianglelefteq G$ is a closed normal subgroup that acts freely on X. Let $\phi : X \to X/H$ be the quotient map. Prove that $\phi^\dagger$ induces an equivalence $\mathrm{MHM}_{G/H}(X/H) \xrightarrow{\sim} \mathrm{MHM}_G(X)$.

Exercise 6.4 (Induction equivalence)**.** Let $H \leq G$ be a closed subgroup, and let X be a variety with an H-action. Prove that there is an equivalence of categories

$$\mathrm{MHM}_H(X) \cong \mathrm{MHM}_G(G \times^H X).$$

Here, $G \times^H X$ is the quotient of $G \times X$ by the free H-action given by $h \cdot (g, x) = (gh^{-1}, hx)$.

Exercise 6.5. Let $X_{G,\mathrm{sm}}$ be the site whose objects are G-equivariant smooth maps $U \to X$. Show that the prestack which assigns to $U \to X$ the category $\mathrm{MHM}_G(U)$ is a stack.

7. Definition of $D^{\mathrm{b}}_G(X)$

Let us define a G-equivariant analogue of the site X_{acyc}. We define $\mathrm{Acyc}_G(X)$ to the be the category whose objects G-equivariant acyclic morphisms $U \xrightarrow{u} X$. A morphism $(U \xrightarrow{u} X) \to (V \xrightarrow{v} X)$ is a G-equivariant acyclic map $f : U \to V$ such that $v \circ f = u$. We define covering families just as in Definition 3.8, obtaining a Grothendieck topology on $\mathrm{Acyc}_G(X)$. The resulting site is denoted $X_{G,\mathrm{acyc}}$.

Definition 7.1. An *n-acyclic resolution* of X is a G-equivariant n-acyclic map $u : U \to X$ such that the G-action on U is free.

Let $\mathrm{Res}_G(X) \subset \mathrm{Acyc}_G(X)$ denote the full subcategory consisting of acyclic resolutions.

Proposition 7.2. *Let X be a smooth G-variety. There exists a covering family in $X_{G,\mathrm{acyc}}$ consisting entirely of objects in* $\mathrm{Res}_G(X)$*.*

Proof. Embed G in some GL_k, and note that if $U \to X$ is an n-acyclic GL_k-equivariant resolution, then it is clearly also an n-acyclic G-equivariant resolution. Thus, we may assume that $G = GL_k$. We may also assume that X is a point, as one can then obtain n-acyclic resolutions for arbitrary X by base change.

Thus, for each $n \geq k$, we seek a smooth variety U_n with a free action of GL_k such that $H^j(U_n) = 0$ for $0 \leq j \leq n$. The complex Stiefel manifold of k-frames in $\mathbb{C}^{n+k}$ is known to have these properties [16, Theorem IV.4.7]. $\square$

Note that $\mathrm{Res}_G(X)$ enjoys the following "stability" property: if $\mathcal{U} \to \mathcal{V}$ is a morphism in $\mathrm{Acyc}_G(X)$, and $\mathcal{V} \in \mathrm{Res}_G(X)$, then $\mathcal{U}$ lies in $\mathrm{Res}_G(X)$ as well. This property makes it possible to define a prestack $\mathsf{pre}\mathbf{D}_{G,X}$ on $X_{G,\mathrm{acyc}}$ as follows:

$$\mathsf{pre}\mathbf{D}_{G,X}(U \xrightarrow{u} X) = \begin{cases} D^{\mathrm{b}}\mathrm{MHM}(U/G) & \text{if } U \xrightarrow{u} X \text{ is a resolution of } X, \\ 0 & \text{otherwise.} \end{cases}$$

If $(U \xrightarrow{u} X) \xrightarrow{f} (V \xrightarrow{v} X)$ is a morphism in $\mathrm{Res}_G(X)$, the corresponding restriction functor in $\mathsf{pre}\mathbf{D}_{G,X}$ is defined to be

$$\bar{f}^\dagger : D^{\mathrm{b}}\mathrm{MHM}(V/G) \to D^{\mathrm{b}}\mathrm{MHM}(U/G),$$

where $\bar{f} : U/G \to V/G$ is the obvious map induced by $f : U \to V$. We also consider the prestack $\mathsf{pre}\mathbf{D}^{[a,b]}_{G,X}$ that is defined similarly, but with $D^{[a,b]}\mathrm{MHM}(U/G)$ in place of $D^{\mathrm{b}}\mathrm{MHM}(U/G)$.

Proposition 7.3. *If G acts freely on X, then the prestacks $\mathsf{pre}\mathbf{D}^{[a,b]}_{G,X}$ and $\mathsf{pre}\mathbf{D}_{G,X}$ are both stacks.*

Note that in this case, we actually have $\mathrm{Acyc}_G(X) = \mathrm{Res}_G(X)$. The proof, which is quite similar to that of Theorem 3.12, is omitted.

Definition 7.4. Let $\mathbf{D}_{G,X}$ (resp. $\mathbf{D}^{[a,b]}_{G,X}$) denote the stackification of the prestack $\mathsf{pre}\mathbf{D}_{G,X}$ (resp. $\mathsf{pre}\mathbf{D}^{[a,b]}_{G,X}$) on $X_{G,\mathrm{acyc}}$. The *G-equivariant derived category* of X, denoted $D^{\mathrm{b}}_G(X)$, is defined to be the category $\mathbf{D}_{G,X}(X \xrightarrow{\mathrm{id}} X)$.

It is easy to see from the definition that if $f : U \to X$ is any G-equivariant acyclic map, then $D^{\mathrm{b}}_G(U)$ is naturally equivalent to $\mathbf{D}_{G,X}(U \xrightarrow{f} X)$. In particular, if $f : U \to X$ is an acyclic resolution, then by Proposition 7.3, we have $\mathbf{D}_{G,X}(U \xrightarrow{f} X) \cong D^{\mathrm{b}}\mathrm{MHM}(U/G)$.

Remark 7.5. Let X be a G-variety. For each positive integer $n \geq 1$, fix an n-acyclic resolution $u_n : U_n \to X$. Then $\{u_n : U_n \to X\}_{n\geq 1}$ is a covering family in $X_{G,\mathrm{acyc}}$. In view of the preceding remarks, one can specify an object $\mathcal{F} \in D^{\mathrm{b}}_G(X)$ by giving the following data:

(1) for each integer $n \geq 1$, an object $\mathcal{F}_n \in D^{\mathrm{b}}\mathrm{MHM}(U_n/G)$.
(2) for any two integers m, n, an isomorphism $\varphi_{nm} : r^{1\dagger}_{nm}\mathcal{F}_n \xrightarrow{\sim} r^{2\dagger}_{nm}\mathcal{F}_m$.

Exercise 7.6. Here is an alternate approach to describing objects in $D^{\mathrm{b}}_G(X)$, closer in spirit to the descriptions in [7, 12] (see especially [12, Definition 4.2.3 or 4.5.1]). First, fix a commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "i_1"] \arrow[drrrrr, bend right, "p_1"'] & V_2 \arrow[r, "i_2"] \arrow[drrrr, "p_2"'] & V_3 \arrow[r, "i_3"] \arrow[drrr, "p_3"] & \cdots & \cdots \arrow[dr] & \\ &&&&& X \end{tikzcd} \tag{7.1}$$

in which each $p_n : V_n \to X$ is an n-acyclic resolution. However, the i_k's are *not* required to be acyclic, or even smooth, so this is not a diagram in $\mathrm{Acyc}_G(X)$. Let $\pi_k : V_k \to V_k/G$ be the quotient map, and let $\bar{\imath}_k : V_k/G \to V_{k+1}/G$ be the map induced by i_k. Following [12, Definition 4.2.3 or 4.5.1], let $D^{\mathrm{b}}_G(X)'$ denote the category whose objects are tuples $(\mathcal{F}_\infty; (\mathcal{F}_k, j_k, \varphi_k)_{k\geq 1})$, where $\mathcal{F}_\infty \in D^{\mathrm{b}}\mathrm{MHM}(X)$, $\mathcal{F}_k \in D^{\mathrm{b}}\mathrm{MHM}(V_k/G)$, and j_k and φ_k are isomorphisms

$$j_k : \bar{\imath}^\dagger_k\mathcal{F}_{k+1} \xrightarrow{\sim} \mathcal{F}_k, \qquad \varphi_k : p^\dagger_k\mathcal{F}_\infty \xrightarrow{\sim} \pi^\dagger_k\mathcal{F}_k,$$

where we write $\bar{\imath}_k^\dagger$ for the (not necessarily exact) functor $\bar{\imath}_k^*[\dim V_k/G - \dim V_{k+1}/G]$. Furthermore, we require that the following diagram commute:

$$\begin{tikzcd} i_k^\dagger p_{k+1}^\dagger \mathcal{F}_\infty \arrow[r, "\sim"] \arrow[d, "\varphi_{k+1}"'] & p_k^\dagger \mathcal{F}_\infty \arrow[d, "\varphi_k"] \\ i_k^\dagger \pi_{k+1}^\dagger \mathcal{F}_{k+1} \arrow[r, "j_k"'] & \pi_k^\dagger \mathcal{F}_k \end{tikzcd} \tag{7.2}$$

Prove that $D_G^{\mathrm{b}}(X)'$ is equivalent to $D_G^{\mathrm{b}}(X)$.

Hint: It is clear that the collection $\{p_n : V_n \to X\}$ is a covering family in $X_{G,\mathrm{acyc}}$. Given an object of $D_G^{\mathrm{b}}(X)'$, use the commutativity of (7.2) to construct the isomorphisms φ_{nm} of objects on $V_n \times_X V_m$ as in Remark 7.5. In this way, we obtain a functor $D_G^{\mathrm{b}}(X)' \to D_G^{\mathrm{b}}(X)$. Going the other way is a bit subtler: one has to make use of descent in the acyclic topology to construct the φ_k's.

8. Basic properties of equivariant derived categories

The theme of this section is this: for almost any property of $D_G^{\mathrm{b}}(X)$ we want to prove or any construction we want to carry out, it is enough to do it on resolutions of X, and then invoke general properties of stackification.

Lemma 8.1. *Suppose $f : U \to X$ is an n-acyclic resolution, and let $a \le b$ be integers with $b - a < n$. Then the obvious functor $D_G^{[a,b]}(X) \to D^{[a,b]}\mathrm{MHM}(U/G)$ is fully faithful.*

Proof. Let us rename the given map $f : U \to X$ to $u_0 : U_0 \to X$, and then include it as a member of a covering family $\{u_i : U_i \to X\}_{i\in I}$ of acyclic resolutions, where each $u_i : U_i \to X$ is at least n-acyclic. By Corollary 3.6, both functors in the diagram below are fully faithful:

$$D^{[a,b]}\mathrm{MHM}(U_i/G) \xrightarrow{r_{ij}^{1\dagger}} D^{[a,b]}\mathrm{MHM}(U_{ij}/G) \xleftarrow{r_{ij}^{2\dagger}} D^{[a,b]}(U_j/G). \tag{8.1}$$

Now, let $\mathcal{F}, \mathcal{G} \in D_G^{[a,b]}(X)$, and for each $i \in I$, let $\mathcal{F}_i, \mathcal{G}_i \in D^{[a,b]}\mathrm{MHM}(U_i/G)$ be the corresponding objects. In view of (8.1), we have isomorphisms

$$\mathrm{Hom}(\mathcal{F}_i,\mathcal{G}_i) \xrightarrow{\sim} \mathrm{Hom}(r_{ij}^{1\dagger}\mathcal{F}_i, r_{ij}^{1\dagger}\mathcal{G}_i) \cong \mathrm{Hom}(r_{ij}^{2\dagger}\mathcal{F}_j, r_{ij}^{2\dagger}\mathcal{G}_j) \xleftarrow{\sim} \mathrm{Hom}(\mathcal{F}_j,\mathcal{G}_j).$$

Since $\mathbf{Hom}(\mathcal{F},\mathcal{G})$ is a sheaf, specifying a global section, i.e., an element $h \in \mathrm{Hom}(\mathcal{F},\mathcal{G})$, is equivalent to giving a compatible system $\{h_i \in \mathrm{Hom}(\mathcal{F}_i,\mathcal{G}_i)\}_{i\in I}$. But the above isomorphisms show that giving such a compatible system is equivalent to giving $h_0 \in \mathrm{Hom}(\mathcal{F}_0,\mathcal{G}_0)$ alone. Thus, $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}_0,\mathcal{G}_0)$, as desired. ☐

Lemma 8.2. *Each $D_G^{[a,b]}(X)$ is naturally equivalent to a full subcategory of $D_G^{\mathrm{b}}(X)$, and $D_G^{\mathrm{b}}(X)$ is the union of all such full subcategories.*

Proof. This result follows from the fact that it holds when X is replaced by any acyclic resolution $U \to X$, in which it reduces to trivial assertions about the categories $D^{[a,b]}\mathrm{MHM}(U/G)$ and $D^{\mathrm{b}}\mathrm{MHM}(U/G)$. ☐

Theorem 8.3. *$D_G^{\mathrm{b}}(X)$ is a triangulated category. It is equipped with a nondegenerate t-structure whose heart is equivalent to* $\mathrm{MHM}_G(X)$.

Proof. The proof that $D^{\mathrm{b}}_G(X)$ is a triangulated category is very similar to [7, Proposition 2.5.2]. Briefly, the idea is that to check any of the axioms on specific objects, one can work inside some subcategory $D^{[a,b]}_G(X)$. Lemma 8.1 lets us transfer the question to some $D^{[a,b]}\mathrm{MHM}(U/G) \subset D^{\mathrm{b}}\mathrm{MHM}(U/G)$, where the axioms are known to hold. We omit further details.

Next, let $D^{\mathrm{b}}_G(X)^{\leq 0}$ (resp. $D^{\mathrm{b}}_G(X)^{\geq 0}$) be the union of all subcategories $D^{[-n,0]}_G(X)$ (resp. $D^{[0,n]}_G(X)$) for $n \geq 0$. Similar methods to those above show that the pair $(D^{\mathrm{b}}_G(X)^{\leq 0}, D^{\mathrm{b}}_G(X)^{\geq 0})$ constitutes a nondegenerate t-structure on $D^{\mathrm{b}}_G(X)$. Its heart is $D^{[0,0]}_G(X)$. It remains to show that this heart is equivalent to $\mathrm{MHM}_G(X)$. Recall from Exercise 6.5 that the assignment $U \mapsto \mathrm{MHM}_G(U)$ defines a stack on $X_{G,\mathrm{sm}}$, and hence also on $X_{G,\mathrm{acyc}}$. By Theorem 6.2, this stack agrees with the prestack $\mathsf{pre}\mathbf{D}^{[0,0]}_{G,X}$ on acyclic resolutions, so it must be equivalent to the stack $\mathbf{D}^{[0,0]}_{G,X}$. In particular, we have $D^{[0,0]}_G(X) \cong \mathrm{MHM}_G(X)$, as desired. □

Proposition 8.4. *There is a t-exact functor of triangulated categories*

$$\mathsf{For} : D^{\mathrm{b}}_G(X) \to D^{\mathrm{b}}\mathrm{MHM}(X)$$

whose restriction to $\mathrm{MHM}_G(X)$ is isomorphic to $\mathsf{For} : \mathrm{MHM}_G(X) \to \mathrm{MHM}(X)$.

Proof. There is a morphism of prestacks $\mathsf{For} : \mathsf{pre}\mathbf{D}_{G,X} \to \mathbf{D}_X$ which associates to any acyclic resolution $U \to X$ the functor $\phi^\dagger : D^{\mathrm{b}}\mathrm{MHM}(U/G) \to D^{\mathrm{b}}\mathrm{MHM}(U)$, where $\phi : U \to U/G$ is the quotient map. The induced morphism of stacks $\mathsf{For} : \mathbf{D}_{G,X} \to \mathbf{D}_X$ has the desired properties. □

Theorem 8.5. *Let X and Y be smooth G-varieties, and let $f : X \to Y$ be a G-equivariant morphism. There are six functors*

$$\begin{aligned} f^*, f^! &: D^{\mathrm{b}}_G(Y) \to D^{\mathrm{b}}_G(X), & R\mathscr{H}om(\cdot,\cdot) &: D^{\mathrm{b}}_G(X)^{\mathrm{op}} \times D^{\mathrm{b}}_G(X) \to D^{\mathrm{b}}_G(X), \\ f_*, f_! &: D^{\mathrm{b}}_G(X) \to D^{\mathrm{b}}_G(Y), & \cdot \otimes^L \cdot &: D^{\mathrm{b}}_G(X) \times D^{\mathrm{b}}_G(X) \to D^{\mathrm{b}}_G(X), \end{aligned}$$

all of which commute with For.

Proof for f^*. The map $f : X \to Y$ induces a morphism of sites $f : X_{G,\mathrm{acyc}} \to Y_{G,\mathrm{acyc}}$. Consider the push-forward stack $f_\#\mathbf{D}_{G,X}$ on $Y_{G,\mathrm{acyc}}$. It is the stackification of the prestack $f_\#\mathsf{pre}\mathbf{D}_{G,X}$. Consider the morphism of prestacks $f^* : \mathsf{pre}\mathbf{D}_{G,Y} \to f_\#\mathsf{pre}\mathbf{D}_{G,X}$ defined as follows: on an acyclic resolution $u : U \to Y$, it gives the functor

$$\tilde{f}^* : D^{\mathrm{b}}\mathrm{MHM}(U/G) \to D^{\mathrm{b}}\mathrm{MHM}((U \times_Y X)/G),$$

where $\tilde{f} : (U \times_Y X)/G \to U/G$ is the map obtained from f by first forming the pullback $\bar{f} : U \times_Y X \to U$, and then passing to the quotient by the G-action. This determines a morphism of stacks $f^* : \mathbf{D}_{G,Y} \to f_\#\mathbf{D}_{G,X}$. Evaluating at $(Y \xrightarrow{\mathrm{id}} Y) \in Y_{G,\mathrm{acyc}}$ gives us a functor $f^* : D^{\mathrm{b}}_G(Y) \to D^{\mathrm{b}}_G(X)$.

To prove that it commutes with For, it is enough to check at the level of prestacks. Referring to the proof of Proposition 8.4 for the construction of For, we see that we must check that for an acyclic resolution $u : U \to Y$ as above, the diagram

$$\begin{tikzcd} D^{\mathrm{b}}\mathrm{MHM}(U/G) \arrow[r, "\tilde{f}^*"] \arrow[d, "\phi^\dagger"'] & D^{\mathrm{b}}\mathrm{MHM}((U \times_Y X)/G) \arrow[d, "\psi^\dagger"] \\ D^{\mathrm{b}}\mathrm{MHM}(U) \arrow[r, "\bar{f}^*"'] & D^{\mathrm{b}}\mathrm{MHM}(U \times_Y X) \end{tikzcd}$$

commutes. But this is clear. (Here, $\phi : U \to U/G$ and $\psi : U \times_Y X \to (U \times_Y X)/G$ are the quotient maps.) □

Exercise 8.6. Prove the rest of Theorem 8.5.

Exercise 8.7. Prove that (f^*, f_*) is an adjoint pair. Same for $(f_!, f^!)$ and $(\cdot \otimes^L \mathcal{F}, R\mathscr{H}om(\mathcal{F}, \cdot))$

The next two exercises are derived versions of Exercises 6.3 and 6.4.

Exercise 8.8. Suppose G acts on X, and suppose that $H \trianglelefteq G$ is a closed normal subgroup that acts freely on X. Let $\phi : X \to X/H$ be the quotient map. Prove that there is a natural equivalence $D^{\mathrm{b}}_{G/H}(X/H) \xrightarrow{\sim} D^{\mathrm{b}}_G(X)$.

Exercise 8.9 (Induction equivalence)**.** Let $H \leq G$ be a closed subgroup, and let X be a variety with an H-action. Prove that there is an equivalence of categories $D^{\mathrm{b}}_H(X) \cong D^{\mathrm{b}}_G(G \times^H X)$.

9. dgMH-rings and dgMH-modules

Definition 9.1. A *graded mixed Hodge structure* is a sequence $V = (V^n)_{n\in\mathbb{Z}}$ where each $V_n \in \mathrm{MHM}(\mathrm{pt})$ and in which $V_n = 0$ for $n \ll 0$. A *morphism of graded mixed Hodge structures* $f : V \to W$ is simply a collection of mormphisms $(f^n : V^n \to W^n)_{n\in\mathbb{Z}}$ in $\mathrm{MHM}(\mathrm{pt})$. The category of graded mixed Hodge structures is denoted $\mathrm{grMHM}(\mathrm{pt})$.

Note that a graded mixed Hodge structure may have infinitely many nonzero components in positive degrees. It is easy to see that $\mathrm{grMHM}(\mathrm{pt})$ is an abelian category. Moreover, it has an obvious tensor structure: for $V, W \in \mathrm{grMHM}(\mathrm{pt})$, we define $V \otimes W$ by

$$(V \otimes W)^n = \bigoplus_{j+k=n} V^j \otimes W^k.$$

This makes it possible to speak of ring objects in $\mathrm{grMHM}(\mathrm{pt})$, and of module objects over those ring objects. Let $[1] : \mathrm{grMHM}(\mathrm{pt}) \to \mathrm{grMHM}(\mathrm{pt})$ be the shift-of-grading functor, given by

$$(V[1])^n = V^{n+1}.$$

Definition 9.2. A *differential-graded mixed Hodge ring*, or *dgMH-ring*, is a ring object $A \in \mathrm{grMHM}(\mathrm{pt})$ equipped with a map $d : A \to A[1]$ (called the *differential* such that $d[1] \circ d = 0$, and such that the following diagram commutes:

$$\begin{tikzcd} A^i \otimes A^j \arrow[r, "\text{mult.}"] \arrow[d, "(d\otimes \mathrm{id}_{A^j}, (-1)^i \mathrm{id}_{A^i}\otimes d)"'] & A^{i+j} \arrow[d, "d"] \\ (A^{i+1} \otimes A^j) \oplus (A^i \otimes A^{j+1}) \arrow[r, "\text{mult.}"'] & A^{i+j+1} \end{tikzcd} \tag{9.1}$$

Here, the arrows labelled "mult." come from the multiplication map $A \otimes A \to A$. (Of course, the diagram above simply encodes the graded Leibniz rule, usually written as $d(xy) = d(x)y + (-1)^{\deg x} x d(y)$.)

If A is a dgMH-ring, a *differential-graded mixed Hodge A-module*, or a *dgMH-A-module*, or a *dgMH-module over A*, is a module object M over A in grMHM(pt) equipped with a differential $d : M \to M[1]$ satisfying $d[1] \circ d = 0$ and a suitable analogue of (9.1).

The notions of *homotopy* and *quasi-isomorphism* for dgMH-modules are defined as usual. We can form the homotopy category $K\mathrm{MH}(A)$ (whose objects of dgMH-modules and whose morphisms are homotopy classes of maps of dgMH-modules) as well as the derived category $D\mathrm{MH}(A)$ (obtained from $K\mathrm{MH}(A)$ by inverting the quasi-isomorphisms). Both of these are, of course, triangulated categories.

Definition 9.3. The *perfect derived category* of a dgMH-ring A, which will be denoted by $D^{\mathrm{pf}}\mathrm{MH}(A)$, is the full triangulated subcategory of $D\mathrm{MH}(A)$ generated by Tate twists of direct summands of the free module A.

9.1. **Affine even stratifications.** In this section, which depends heavily on the ideas in [3], we show how to describe the derived category of certain stratified spaces in terms of dgMH-modules. Let X be a smooth variety equipped with an algebraic stratification $X = \bigcup_{s \in S} X_s$. For each $s \in S$, let $\mathscr{L}_s \in \mathrm{MHM}(X)$ denote the simple mixed Hodge module associated to the trivial mixed Hodge module on X_s. (Thus, $\mathrm{rat}(\mathscr{L}_s)$ is the intersection cohomology complex on $\overline{X_s}$, and $\mathscr{L}_s$ is pure of weight $\dim X_s$.) We call this an *affine even stratification* if the following conditions hold (see [3, §3.4]:

(1) Each X_s is isomorphic to an affine space.
(2) $H^i(\mathscr{L}_s|_{X_t}) \in \mathrm{MHM}(X_t)$ vanishes if $i \not\equiv \dim X_s - \dim X_t \pmod 2$, and is isomorphic to a direct sum of copies of $\underline{\mathbf{A}}_{X_t}((\dim X_t - \dim X_s - i)/2)$ otherwise.

The main example [3, Corollary 3.3] is that of a partial flag variety of a reductive group, equipped with the Schubert stratification.

In this section, it will be convenient to formally introduce a "square-root of the Tate twist." See the discussion following [6, Definition 4.1.4] or [3, §6.1] for an explanation of this. This operation involves replacing $\mathrm{MHM}(X)$ and $D^{\mathrm{b}}\mathrm{MHM}(X)$ by new, larger categories, but in a minor abuse of notation, we will continue to write $\mathrm{MHM}(X)$ and $D^{\mathrm{b}}\mathrm{MHM}(X)$ for the new categories in which "$(\frac{1}{2})$" makes sense.

Throughout this section, we write $\mathrm{MHM}_S(X) \subset \mathrm{MHM}(X)$ (resp. $D^{\mathrm{b}}_S\mathrm{MHM}(X) \subset D^{\mathrm{b}}\mathrm{MHM}(X)$) for the Serre (resp. triangulated) subcategory generated by the objects $\{\mathscr{L}_s(\frac{n}{2})\}_{s \in S, n \in \mathbb{Z}}$. Let $L = \bigoplus \mathscr{L}_s(-\frac{\dim X_s}{2})$. This is the direct sum of all simple objects of weight 0. Finally, let E be the dgMH-ring given by

$$E^i = \underline{\mathrm{Hom}}^i(L, L),$$

with zero differential. Our goal is to describe $D^{\mathrm{b}}_S\mathrm{MHM}(X)$ in terms of dgMH-modules over E. Let P_s be the dgMH-module given by

$$P_s^i = \underline{\mathrm{Hom}}^i(L, \mathscr{L}_s(-\tfrac{\dim X_s}{2})),$$

again with zero differential. Clearly, the direct sum $\bigoplus_{s \in S} P_s$ is isomorphic to the free module E. Moreover, each P_s is indecomposable. Thus, $D^{\mathrm{pf}}\mathrm{MH}(E)$ is the subcategory of $D\mathrm{MH}(E)$ generated by Tate twists of the P_s's.

Theorem 9.4. *Let X, $\{X_s\}_{s\in S}$, L, E, and $\{P_s\}_{s\in S}$ be as above. There is an equivalence of categories*

$$\alpha : D^{\mathrm{b}}_S\mathrm{MHM}(X) \xrightarrow{\sim} D^{\mathrm{pf}}\mathrm{MH}(E)$$

such that $\alpha(\mathscr{L}_s) \cong P_s(\frac{\dim X_s}{2})$.

Proof Sketch. Recall [6, Theorem 3.3.1] that $\mathrm{Perv}_S(X)$ has enough projectives. We will call an object of $\mathrm{MHM}_S(X)$ "rat-projective" if its image under rat is a projective object of $\mathrm{Perv}_S(X)$. Now, $\mathrm{MHM}_S(X)$ does not have enough projectives, but it does have enough rat-projectives. Indeed, as explained in [3, §6.2], it is possible to find a finite rat-projective resolution $P^\bullet \to L$ which has the additional property of being *linear*, meaning that every simple quotient of a given term P^{-i} in the resolution is pure of weight $-i$. Let A be the dgMH-ring given by

$$A^i = \bigoplus_{j+k=i} \underline{\mathrm{Hom}}(P^{-j}, P^k),$$

and with differential induced by that of the complex $P^\bullet$.

It is easy to see that the cohomology of A is isomorphic to E. Moreover, as explained in [3, Lemma 6.5], there is an equivalence of categories

$$D^{\mathrm{pf}}\mathrm{MH}(E) \xrightarrow{\sim} D^{\mathrm{pf}}\mathrm{MH}(A). \tag{9.2}$$

(To be precise, [3, Lemma 6.5] actually involves ordinary dg-rings and dg-modules over a field, rather than for ring and module objects in grMHM(pt), but it is straightforward to adapt its proof to our setting.)

Finally, one can define a functor $D^{\mathrm{b}}_S\mathrm{MHM}(X) \to D^{\mathrm{pf}}\mathrm{MH}(A)$ by the formula $\mathcal{F}^\bullet \mapsto \underline{\mathrm{Hom}}^\bullet(P^\bullet, \mathcal{F}^\bullet)$. Moreover, this functor is an equivalence of categories, for the reasons sketched, e.g., in [3, Lemma 6.2] and the preceding comments. Combining this functor with (9.2) yields the desired equivalence. □

Remark 9.5. The functor in (9.2) is given by tensoring with a suitable bimodule, and *not* by a dgMH-ring homomorphism. A natural question is whether

It is natural to ask whether A and E are quasi-isomorphic; in other words, whether A is formal. It is shown in [3, Corollary 6.7] that the ordinary dg-ring $\mathrm{rat}(A)$ is formal, but the argument there does not seem to lift to grMHM(pt).

Proposition 9.6. *Let X be a smooth connected variety, and let $D^{\mathrm{b}}_{\mathrm{const}}\mathrm{MHM}(X) \subset D^{\mathrm{b}}\mathrm{MHM}(X)$ denote the full triangulated subcategory generated by $\{\underline{\mathbf{A}}_X(n)\}_{n\in\mathbb{Z}}$. Assume that X admits an affine even stratification. Then there is an equivalence of categories*

$$\alpha : D^{\mathrm{b}}_{\mathrm{const}}\mathrm{MHM}(X) \xrightarrow{\sim} D^{\mathrm{pf}}\mathrm{MH}(H^\bullet(X)).$$

Here, $H^\bullet(X) \in$ grMHM(pt) is the cohomology ring of X, regarded as a dgMH-ring with zero differential.

Proof Sketch. Retain the notation of the Theorem 9.4. Because X is connected, there must be a unique open stratum X_o in the given affine even stratification, and then, because X is smooth, we have $\mathscr{L}_o \cong \underline{\mathbf{A}}_X$. Thus, $D^{\mathrm{b}}_{\mathrm{const}}\mathrm{MHM}(X)$ is the full triangulated subcategory of $D^{\mathrm{b}}_S\mathrm{MHM}(X)$ generated by $\mathscr{L}_o$, and so it is equivalent to the full triangulated subcategory of $D^{\mathrm{pf}}\mathrm{MH}(E)$ generated by P_o. Let us denote the latter category by $D^{\mathrm{pf}}_{P_o}\mathrm{MH}(E)$. Let H be the dgMH-ring $\underline{\mathrm{Hom}}^\bullet(P_o, P_o)$ (with zero differential). On the one hand, since $H^\bullet(X) \cong \underline{\mathrm{Hom}}^\bullet(\mathscr{L}_o, \mathscr{L}_o)$, Theorem 9.4 gives us a natural isomorphism $H^\bullet(X) \cong H$. On the other hand, one can show that P_o is a projective E-module, and then standard ring-theoretic arguments show that $D^{\mathrm{pf}}_{P_o}\mathrm{MH}(E) \cong D^{\mathrm{pf}}\mathrm{MH}(H)$, as desired. $\square$

10. Equivariant derived category of a point

Theorem 10.1. *Assume that G is a torus. There is an equivalence of categories $D^{\mathrm{b}}_G(\mathrm{pt}) \cong D^{\mathrm{pf}}\mathrm{MH}(H^\bullet_G(\mathrm{pt}))$.*

Proof. For brevity, let $A = H^\bullet_G(\mathrm{pt})$. Observe first that $D^{\mathrm{pf}}\mathrm{MH}(A)$ admits a unique t-structure containing all $A(n)$ in its heart. Let $D^{[a,b]}\mathrm{MH}(A) \subset D^{\mathrm{pf}}\mathrm{MH}(A)$ be the full subcategory consisting of objects M whose cohomology with respect to that t-structure ${}^tH^i(M)$ vanishes unless $a \le i \le b$.

Let us first show that $D^{[a,b]}_G(\mathrm{pt}) \cong D^{[a,b]}\mathrm{MH}(A)$ for any interval $[a,b]$. Let $d = \dim G$, and fix an isomorphism $G \cong \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$. Let $U_n = (\mathbb{C}^{n+1} \smallsetminus \{0\}) \times \cdots \times (\mathbb{C}^{n+1} \smallsetminus \{0\})$ (d copies). There is an obvious free action of G on U_n, and the quotient is given by

$$U_n/G \cong \mathbb{P}^n \times \cdots \times \mathbb{P}^n.$$

It is clear that $U_n \to \mathrm{pt}$ is an $2n$-acyclic resolution. By Lemma 8.1, if $b - a < 2n$, then $D^{[a,b]}_G(\mathrm{pt})$ is equivalent to a full subcategory of $D^{[a,b]}\mathrm{MHM}((\mathbb{P}^n)^d)$. Indeed, tracing through the definitions, one can check that $D^{[a,b]}_G(\mathrm{pt}) \cong D^{[a,b]}_{\mathrm{const}}\mathrm{MHM}((\mathbb{P}^n)^d)$. Now, $(\mathbb{P}^n)^d$ admits an affine even stratification, so by Proposition 9.6, we have $D^{[a,b]}_G(\mathrm{pt}) \cong D^{[a,b]}\mathrm{MH}(H^\bullet((\mathbb{P}^n)^d)$. (The latter is defined in the same way as $D^{[a,b]}\mathrm{MH}(A)$ above.)

The underlying graded ring of $H^\bullet((\mathbb{P}^n)^d)$ (i.e., forgetting the mixed Hodge structure) is $\mathbf{A}[x_1, \ldots, x_d]/(x_1^n, \ldots, x_d^n)$, where each x_i has degree 2. As an object of grMHM(pt), its component in degree $2k$ is isomorphic to a direct sum of copies of $\underline{\mathbf{A}}(-k)$. The cohomology $H^\bullet_G(\mathrm{pt})$ has a similar description, except that its underlying graded ring is simply $\mathbf{A}[x_1, \ldots, x_d]$. From these descriptions, it is clear that $D^{[a,b]}\mathrm{MH}(H^\bullet((\mathbb{P}^n)^d) \cong D^{[a,b]}\mathrm{MH}(A)$ if $b - a < 2n$. We conclude that

$$D^{[a,b]}_G(\mathrm{pt}) \cong D^{[a,b]}\mathrm{MH}(A). \tag{10.1}$$

Now, $D^{\mathrm{b}}_G(\mathrm{pt})$ is the union of all $D^{[a,b]}_G(\mathrm{pt})$, and likewise for the right-hand side. Thus, to finish the proof, it suffices to check that the equivalence (10.1) is compatible with the inclusions $D^{[a,b]}_G(\mathrm{pt}) \to D^{[a',b']}_G(\mathrm{pt})$, where $[a,b] \subset [a',b']$. This compatibility is left to the reader. $\square$

11. Appendix: Quasi-equivariant $\mathscr{D}$-modules

As noted earlier, the development of $D^{\mathrm{b}}_G(X)$ in this lecture is quite close to that of the constructible equivariant derived category in [7], or of the equivariant derived category of $\mathscr{D}$-modules in [12].

But in the $\mathscr{D}$-module setting, there is another possibility: one can consider the category whose objects are $\mathscr{D}$-modules together with an equivariant structure on the underlying $\mathscr{O}$-module. Such an object is called a *quasi-equivariant $\mathscr{O}$-module.* A precise definition and a number of basic results appear in [12, Chap. 3]. We denote the category of quasi-G-equivariant $\mathscr{D}$-modules by $\mathrm{Mod}(\mathscr{D}_X, G)$.

Remarkably, the naïve derived category $D^{\mathrm{b}}\mathrm{Mod}(\mathscr{D}_X, G)$ is the "correct" setting for derived functors of quasi-equivariant $\mathscr{D}$-modules: as shown in [12, SS3.4–3.8], it is possible to define all the usual sheaf operations on this category in such a way that they commute with the forgetful functor $D^{\mathrm{b}}\mathrm{Mod}(\mathscr{D}_X, G) \to D^{\mathrm{b}}\mathrm{Mod}(\mathscr{D}_X)$. Quasi-equivariant $\mathscr{D}$-modules essentially inherit this behavior from equivariant $\mathscr{O}$-modules.

A brief treatment of the category of equivariant (quasicoherent) $\mathscr{O}$-modules, denoted $\mathrm{Mod}_G(\mathscr{O}_X)$, appears, for instance, in [8] or [1]. Assume for simplicity that G is connected. In that case, the reason that $D^{\mathrm{b}}\mathrm{Mod}_G(\mathscr{O}_X)$ is the correct derived category to work in can be summarized quite succinctly: the coherent cohomology of G (like that of any affine variety) vanishes in positive degrees, so the action map $a : G \times X \to X$ is always a "coherently ∞-acyclic resolution." Using the fact that every smooth admits a coherently ∞-acyclic resolution, one can show that the assignment $U \mapsto D^{\mathrm{b}}\mathrm{Mod}_G(U)$ defines a stack on the $\mathscr{O}$-module analogue of $X_{G,\mathrm{acyc}}$.

It is natural to ask whether these developments can be imitated in the mixed Hodge module setting.

Definition 11.1. A *quasi-G-equivariant mixed Hodge module* consists of an object $\mathcal{F} \in \mathrm{MHM}(X)$ together with a quasi-equivariant structure on the underlying $\mathscr{D}$-module of $\mathcal{F}$. The category of quasi-G-equivariant mixed Hodge modules is denoted $\mathrm{MHM}(X, G)$.

Question 11.2. Let X be a smooth G-variety. On $X_{G,\mathrm{acyc}}$, does the assignment $U \mapsto D^{\mathrm{b}}\mathrm{MHM}(X, G)$ define a stack? If not, what is its stackification?

I do not know the answers to these questions. For the first question, it seems unlikely that the $\mathscr{D}$-module arguments of [12] can be adapted to the mixed Hodge module setting, because those arguments require one to work in the category of *all* $\mathscr{D}$-modules, not just regular holonomic ones. On the other hand, I do not understand the category $\mathrm{MHM}(X, G)$ well enough to exhibit a counterexample.

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