INTRODUCTION TO STAGGERED SHEAVES

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Abstract. This note is an expository account of the theory of staggered sheaves, based on a series of lectures given by the author at RIMS (Kyoto) in October 2008.

Perverse sheaves have become a tool of great importance in representation theory, largely because of the remarkable way in which they provide a link between geometry and algebra. A single perverse sheaf contains information reminiscent of classical algebraic topology; indeed, the prototypical example of a perverse sheaf comes from the Goresky–MacPherson theory of "intersection homology." But the category of perverse sheaves behaves like the module categories typically seen in representation theory, and some of the most important theorems here are Ext-vanishing and complete reducibility criteria.

Staggered sheaves, the subject of the present note, were introduced by the author in [A1] and subsequently studied in a series of papers [AT1, AT2, A2] by the author and D. Treumann. The category of staggered sheaves also enjoys a long list of remarkable algebraic properties, resembling the most important properties of perverse sheaves. Most notably, the category of staggered sheaves is quasi-hereditary and exhibits "purity" and "decomposition" phenomena. But whereas perverse sheaves are built out of local systems, staggered sheaves are built out of vector bundles, and they live in the derived category of coherent sheaves.

This note is a self-contained exposition of the main results on staggered sheaves. It is meant to be accessible to anyone familiar with the basic theory of algebraic groups and a passing acquaintance with derived categories. Some familiarity with perverse sheaves may be helpful for motivation, but the relevant facts will be reviewed in Section 2. Most proofs will be omitted. No new results appear here.

Notation and definitions occupy Section 3, and the main theorems are listed in Sections 4–6. An example on $\mathbb{P}^1$ is worked out in Section 7, and some open questions are posed in Section 8. Appendix A surveys the theory of *baric structures*, required for the proofs of the main theorems, and Appendix B describes differences in terminology, notation, and conventions among the various papers.

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1. Prologue: Staggered Representations

In this section, we carry out a construction that is parallel to, and exhibits the main features of, that of staggered sheaves. Sheaves do not explicitly appear in this section, but the ideas in this section will be revisited later.

Let H be an algebraic group over an algebraically closed field $\Bbbk$. Assume for simplicity that either $\operatorname{char}\Bbbk = 0$, or else that H is solvable. In either case, H admits a Levi decomposition $H = LU$, where L is reductive, and U is the unipotent radical of H. Moreover, the rational representations of L are all semisimple. (If H is solvable, L is a torus.) Let $\mathfrak{Rep}(H)$ denote the category of finite-dimensional rational representations of H, and let $\operatorname{Irr}(H)$ denote the set of (isomorphism classes of) irreducible rational representations.

We now describe a recipe for attaching an integer, called the *step*, to each $V \in \operatorname{Irr}(H)$. Choose, once and for all, a cocharacter

$$\xi : \mathbb{G}_{\mathrm{m}} \to \mathrm{Z}^\circ(L),$$

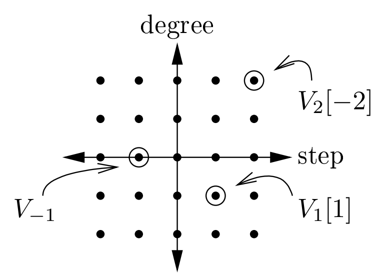
where $\mathrm{Z}^\circ(L)$ denotes the identity component of the center of L, and assume that

$$\langle \xi, \lambda \rangle < 0 \text{ for all weights } \lambda \text{ of } \mathrm{Z}^\circ(L) \text{ on } \operatorname{Lie}(U). \tag{1.1}$$

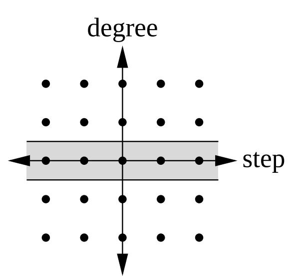
Next, recall that U acts trivially on any irreducible representation of H, so $\operatorname{Irr}(H)$ can be identified with $\operatorname{Irr}(L)$. In any irreducible representation V of L, the subgroup $\mathrm{Z}^\circ(L)$ acts by a character $\chi_V : \mathrm{Z}^\circ(L) \to \mathbb{G}_{\mathrm{m}}$. We define the step of V by

$$\operatorname{step} V = \langle \xi, \chi_V \rangle.$$

Next, let $\mathcal{D} = D^b(\mathfrak{Rep}(H))$ denote the bounded derived category of $\mathfrak{Rep}(H)$. We regard $\mathfrak{Rep}(H)$ as a subcategory of $\mathcal{D}$ as usual, by thinking of objects of $\mathfrak{Rep}(H)$ as chain complexes concentrated in degree 0. Objects of the form $V[n]$, where $V \in \operatorname{Irr}(H)$, may be plotted on a two-dimensional grid whose horizontal axis indicates step, and whose vertical axis indicates cohomological degree in $\mathcal{D}$:



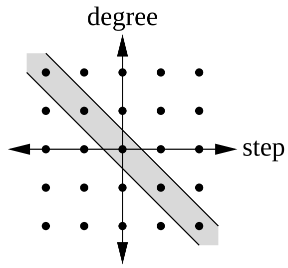
Of course, not all objects in $\mathcal{D}$ live on points on this grid, but $\mathcal{D}$ is generated by such objects: all chain complexes are built up by extensions from complexes concentrated in a single degree, and all H-representations are built up from irreducible representations. Indeed, the subcategory $\mathfrak{Rep}(H)$ may be pictured thus:



Let $\mathcal{M}(H)$ denote the subcategory generated by objects of the form

$$V[\operatorname{step} V], \qquad \text{where } V \in \operatorname{Irr}(H).$$

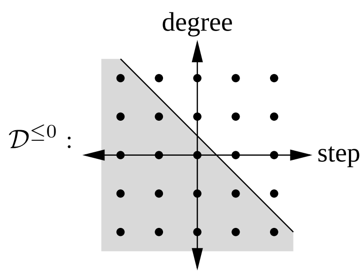
In our picture of $\mathcal{D}$, $\mathcal{M}(H)$ is generated by objects lying along a line of slope -1:

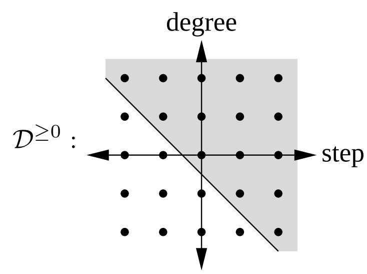


Theorem 1.1. *$\mathcal{M}(H)$ is a semisimple abelian subcategory of $\mathcal{D}$. The simple objects are precisely the objects of the form $V[\operatorname{step} V]$, with $V \in \operatorname{Irr}(H)$.*

Proof sketch. The usual way to produce abelian subcategories of triangulated categories is to use the machinery of *t-structures* [BBD]. A t-structure on $\mathcal{D}$ is pair of subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, satisfying a short list of axioms that will be recalled below.

In our case, we define $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$) to be the subcategory generated by objects $V[n]$ with $V \in \operatorname{Irr}(H)$ and $n \geq \operatorname{step} V$ (resp. $n \leq \operatorname{step} V$). Thus:





The three axioms to be checked are as follows:

(1) *$\mathcal{D}^{\leq 0}[1] \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq 0}[1]$.* This is obvious from the definition.

(2) *$\operatorname{Hom}(A, B[-1]) = 0$ for all $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 0}$.* By a standard induction argument, this can be reduced to showing for any $V, W \in \operatorname{Irr}(H)$, we have $\operatorname{Hom}(V[n], W[m]) = 0$ whenever $n \geq \operatorname{step} V$ and $m < \operatorname{step} W$. Equivalently, we must show that $\operatorname{Ext}^{m-n}(V, W)$ vanishes. If $m - n < 0$, this is trivial, but if $m - n \geq 0$, this must be proved using the assumption (1.1).

(3) *For any object $X \in \mathcal{D}$, there is a distinguished triangle $A \to X \to B \to$ with $A \in \mathcal{D}^{\leq 0}$ and $B[1] \in \mathcal{D}^{\geq 0}$.* The proof of this proceeds by induction on the number of nonzero cohomology objects of X and a diagram-chasing argument using the 9-lemma [BBD, Proposition 1.1.11].

The proof of the semisimplicity of $\mathcal{M}(H)$ and the determination of simple objects both come down to Hom-group calculations similar to those in part (2) above. $\square$

2. Review of Perverse Sheaves and Weights

In this section, we briefly review the most important facts about perverse sheaves, at least from the viewpoint of applications in representation theory. Subsequent sections do not depend on any facts listed here; rather, the list is meant to serve as motivation for later results on staggered sheaves.

Let X be a variety over the algebraic closure $\overline{\mathbb{F}}_q$ of a finite field $\mathbb{F}_q$. Suppose X is endowed with a fixed stratification $\mathcal{S}$ into finitely many locally closed smooth strata, as well as a finite collection $\mathcal{L}$ of isomorphism classes of local systems on those strata. We assume that the pair $(S, \mathcal{L})$ satisfies the conditions of [BBD, §2.2.10(a)–(c)], which are technical conditions meant to ensure that the usual sheaf functors behave well.

Fix a prime number ℓ different from the characteristic of $\mathbb{F}_q$, and let $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ denote the "derived" category of bounded mixed constructible complexes of étale $\bar{\mathbb{Q}}_\ell$-sheaves. (In the presence of a group action, one should instead take $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}()$ to denote the Bernstein–Lunts equivariant derived category [BL].) For each stratum $S \in \mathcal{S}$, let $i_S : S \hookrightarrow X$ denote the inclusion map. Next, we define two new full subcategories of $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ as follows:

$$
\begin{aligned}
{}^p\mathrm{D}^{\leq 0} &= \{F \mid h^k(i_S^* F) = 0 \text{ if } k > -\dim S \text{ for all } S \in \mathcal{S}\}, \\
{}^p\mathrm{D}^{\geq 0} &= \{F \mid h^k(i_S^! F) = 0 \text{ if } k < -\dim S \text{ for all } S \in \mathcal{S}\}.
\end{aligned}
$$

The category of *perverse sheaves*, denoted $\mathsf{P}(X)$, is defined as follows:

$$\mathsf{P}(X) = {}^p\mathrm{D}^{\leq 0} \cap {}^p\mathrm{D}^{\geq 0}.$$

We begin with a few basic facts about $\mathsf{P}(X)$.

(2.1) The pair $(\mathrm{D}^{\leq 0}, \mathrm{D}^{\geq 0})$ is a nondegenerate, bounded t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$, and therefore $\mathsf{P}(X)$ is an abelian category.

(2.2) Every perverse sheaf has finite length, and there is a bijection

$$\left\{ \begin{array}{c} \text{simple objects} \\ \text{in } \mathsf{P}(X) \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ (C, L) \,\middle|\, \begin{array}{c} C \in \mathbb{O}(X), \text{ and } L \text{ an irreducible} \\ \text{local system on } C \end{array} \right\}$$

The simple object corresponding to a pair (C, L), denoted $\mathrm{IC}(C, L)$, is supported on $\overline{C}$, and its restriction to C is $L[\dim C]$.

(2.3) **Poincaré–Verdier Duality**: The Verdier duality functor $\mathsf{D} : \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ preserves $\mathsf{P}(X)$, and

$$\mathsf{D}\mathrm{IC}(C, L) \simeq \mathrm{IC}(C, L^*),$$

where L^* is the dual local system to L.

Next, we turn to the theory of weights. Recall that being *mixed* means, roughly, that the Frobenius morphism acts on stalks at $\mathbb{F}_q$-points of X with well-behaved eigenvalues. Specifically: all eigenvalues of Frobenius are algebraic integers whose complex absolute values (under any embedding $\overline{\mathbb{Q}} \hookrightarrow C$) are of the form $q^{w/2}$ with $w \in \mathbb{Z}$. When F is actually a sheaf (and not a complex), the integers w that occur are called the *weights* of F.

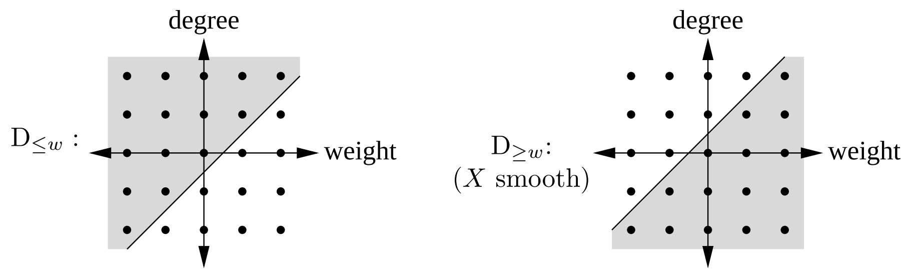
The definition of weights for general objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X)$ is slightly different. The full subcategory of objects of weight $\leq w$, denoted $\mathrm{D}_{\leq w}$, is defined by

$$\mathrm{D}_{\leq w} = \{F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(X) \mid \text{for each } k,\ h^k(F) \text{ has weights } \leq w + k\}.$$

There is also a category $\mathrm{D}_{\geq w}$ of objects of weight $\geq w$. When X is smooth, it can be defined by simply reversing the inequalities above, but in general, that is not the correct definition. Instead, it is defined by

$$\mathrm{D}_{\geq w} = \mathsf{D}(\mathrm{D}_{\leq -w}). \tag{2.4}$$

By plotting sheaf weights along the horizontal axis, we may draw a picture of $D_{\leq w}$ as shown below, in the spirit of the pictures in Section 1. If X is smooth, we may do likewise for $D_{\geq w}$.



However, the latter picture can be misleading for nonsmooth X.

Finally, an object $F \in D^{b}_{m}(X)$ is *pure* of weight w if

$$F \in D_{\leq w} \cap D_{\geq w}.$$

Some important properties of the weight categories $D_{\leq w}$ and $D_{\geq w}$ are as follows. If X is smooth, the constant sheaf $\bar{\mathbb{Q}}_\ell$ on X is pure of weight 0, but that does not hold for general X.

(2.5) $D_{\leq w}[1] = D_{\leq w+1}$ and $D_{\geq w}[1] = D_{\geq w+1}$.

(2.6) If $F \in D_{\leq w}$ and $G \in D_{\geq w+1}$, then $\mathrm{Hom}^i(F, G) = 0$ for all $i > 0$.

(2.7) (Deligne [D]) For a proper morphism $f : X \to Y$, the functor $f_! = f_*$ takes pure objects to pure objects.

The last item is part of Deligne's proof of the Weil conjectures. Specifically, when X is a smooth projective variety, it follows from (2.7) (by taking Y to be a point) that each ℓ-adic cohomology group $H^k(X, \bar{\mathbb{Q}}_\ell)$ is pure of weight k.

(2.8) If F and G are perverse sheaves with $F \in D_{\leq w}$ and $G \in D_{\geq w+1}$, then in fact $\mathrm{Hom}^i(F, G) = 0$ for all $i \geq 0$.

(2.9) **Purity:** Every perverse sheaf F admits a canonical finite filtration

$$\cdots \subset F_{-1} \subset F_0 \subset F_1 \subset \cdots$$

such that F_w/F_{w-1} is pure of weight w. In particular, every simple perverse sheaf is pure.

(2.10) **Decomposition:** Any pure object in $D^{b}_{m}(X)$ is a direct sum of shifts of simple perverse sheaves.

For applications in representation theory, this Decomposition Theorem is usually used in conjunction with (2.7).

Finally, we consider "standard" and "costandard" objects. Precise definitions will be given in Section 6; for now, one may keep in mind that Verma modules are standard objects in category $\mathcal{O}$ for a complex semisimple Lie algebra. Indeed, Verma modules are perhaps the motivating example for the general notion.

(2.11) Assume that each stratum S is isomorphic to an affine space $\mathbb{A}^k$. Then every simple perverse sheaf has a standard cover and a costandard hull. Moreover, $\mathsf{P}(X)$ has enough projectives and enough injectives.

3. Preliminaries

3.1. **Notation and assumptions.** Let $\Bbbk$ be an algebraically closed field. We temporarily assume that $\operatorname{char} \Bbbk = 0$ (this will soon be dropped). Let us put:

G	a reductive algebraic group over $\Bbbk$
X	a variety over $\Bbbk$ on which G acts with finitely many orbits
$\mathbb{O}(X)$	the set of G-orbits in X
$\mathcal{C}_G(X)$	the category of G-equivariant coherent sheaves on X
$\mathcal{D}^{\mathrm{b}}_G(X)$	the bounded derived category of $\mathcal{C}_G(X)$
ω_X	the equivariant dualizing complex of X in $\mathcal{D}^{\mathrm{b}}_G(X)$

To be more precise, we normalize ω_X in the following way: If X is smooth, then

$$\omega_X \simeq \Omega_X^{\mathrm{top}}[\dim X],$$

where Ω_X^{top} denotes the canonical bundle of X. If X is not smooth, the Sumihiro embedding theorem [S] tells us that we may find a closed G-equivariant embedding $\iota : X \hookrightarrow Y$ into a smooth G-variety Y. We then set

$$\omega_X = R\iota^! \omega_Y.$$

This object is independent of the choice of ι.

We will also require the bounded-above and bounded-below derived categories $\mathcal{D}^-_G(X)$ and $\mathcal{D}^+_G(X)$. Recall [H] that the latter must be defined to consist of bounded-below chain complexes of *quasicoherent* sheaves with coherent cohomology.

For each orbit $C \in \mathbb{O}(X)$, we define the following:

$i_C : C \hookrightarrow X$	the inclusion of C as a reduced locally closed subscheme
H_C	the G-stabilizer of some point of C
$H_C = L_C U_C$	a Levi decomposition of H_C
$\mathrm{Z}^\circ(L_C)$	the identity component of the center of the Levi factor of H_C
$\mathfrak{u}_C$	the Lie algebra of the unipotent radical of H_C
$\mathcal{I}_C$	the ideal sheaf in $\mathcal{O}_X$ for the closed subvariety $\overline{C}$
$N^*(C)$	conormal bundle of $C \subset X$

Above, the assumption that $\operatorname{char} \Bbbk = 0$ was used in two places: the invocation of the Sumihiro embedding theorem for the definition of ω_X, and the existence of a Levi decomposition for H_C. We would now like to drop the assumption that $\operatorname{char} \Bbbk = 0$, so if $\operatorname{char} \Bbbk \neq 0$, we explicitly impose the following additional assumptions

(3.1) X admits a closed embedding $\iota : X \hookrightarrow Y$ into a smooth G-variety Y.

(3.2) Each H_C is solvable.

Condition (3.1) is really not essential (see Section B.3), but in the absence of (3.2), the Decomposition Theorem (Theorem 5.6) fails. Both hold in the important example where G is a Borel subgroup of reductive group and X is a flag variety.

With these assumptions, we now have in any characteristic:

(3.3) All representations of the reductive groups L_C are completely reducible.

Recall that on a single G-orbit C, every equivariant coherent sheaf is necessarily locally free. In other words, $\mathcal{C}_G(C)$ is in fact the category of G-equivariant vector bundles on C, which in turn is equivalent to the category $\mathfrak{Rep}(H_C)$ of rational representations of the isotropy group H_C. We will freely make use of the equivalence

$$\mathcal{C}_G(C) \simeq \mathfrak{Rep}(H_C)$$

in the sequel. In particular, the Lie algebra $\mathfrak{u}_C$ equipped with the adjoint action of H_C may be regarded as a coherent sheaf on C, and the vector bundle $N^*(C)$ may be regarded as an H_C representation.

3.2. **Definition of staggered sheaves.** The category of staggered sheaves depends on the following choices:

(3.4) For each orbit $C \in \mathbb{O}(X)$, a cocharacter $\xi_C : \mathbb{G}_{\mathrm{m}} \to \mathrm{Z}^\circ(L_C)$ such that

$$\langle \xi_C, \lambda \rangle \leq -1 \qquad \text{for all weights } \lambda \text{ of } \mathrm{Z}^\circ(L_C) \text{ on } \mathfrak{u}_C \text{ and on } N^*(C).$$

Such a collection $\{\xi_C \mid C \in \mathbb{O}(X)\}$ is called an *s-structure* on X.

(3.5) A function $r : \mathbb{O}(X) \to \mathbb{Z}$. This will be known as the *perversity function*.

Fix an s-structure and a perversity once and for all.

Definition 3.1. Let $\mathcal{F} \in \mathcal{C}_G(C)$ be an irreducible vector bundle, and let $\chi_{\mathcal{F}} : \mathrm{Z}^\circ(L_C) \to \mathbb{G}_{\mathrm{m}}$ be the character of $\mathrm{Z}^\circ(L_C)$ on the corresponding H_C-representation. The *step* of $\mathcal{F}$ is defined by

$$\operatorname{step} \mathcal{F} = \langle \xi_C, \chi_{\mathcal{F}} \rangle.$$

The notion of *step* does not make sense for general objects of $\mathcal{C}_G(C)$. Nevertheless, we can form filtrations of $\mathcal{C}_G(C)$ in terms of steps of irreducible objects. For $w \in \mathbb{Z}$, we define full subcategories of $\mathcal{C}_G(C)$ as follows:

$$\mathcal{C}_G(C)_{\leq w} = \{\mathcal{F} \mid \text{for every irreducible subquotient } \mathcal{G} \text{ of } \mathcal{F}, \operatorname{step} \mathcal{G} \leq w\},$$
$$\mathcal{C}_G(C)_{\geq w} = \{\mathcal{F} \mid \text{for every irreducible subquotient } \mathcal{G} \text{ of } \mathcal{F}, \operatorname{step} \mathcal{G} \geq w\}.$$

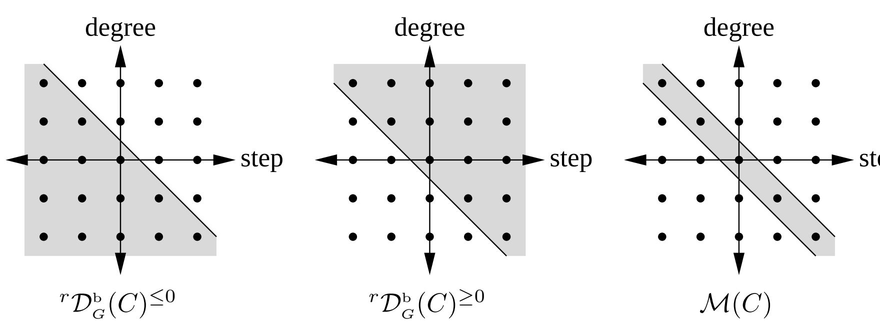
Using these, we define two full subcategories of $\mathcal{D}^{\mathrm{b}}_G(X)$ as follows:

$${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0} = \{\mathcal{F} \mid h^k(Li^*_C\mathcal{F}) \in \mathcal{C}_G(C)_{\leq r(C)-k} \text{ for all } C \in \mathbb{O}(X) \text{ and all } k \in \mathbb{Z}\},$$
$${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\geq 0} = \{\mathcal{F} \mid h^k(Ri^!_C\mathcal{F}) \in \mathcal{C}_G(C)_{\geq r(C)-k} \text{ for all } C \in \mathbb{O}(X) \text{ and all } k \in \mathbb{Z}\}.$$

(The unbounded versions ${}^r\mathcal{D}^-_G(X)^{\leq 0}$ and ${}^r\mathcal{D}^+_G(X)^{\geq 0}$ are defined similarly.)

Definition 3.2. The category of *staggered sheaves* on X, denoted ${}^r\mathcal{M}(X)$, or simply $\mathcal{M}(X)$, is the category ${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0} \cap {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\geq 0}$.

It is clear than when X consists of a single orbit C, the category $\mathcal{M}(C)$ is equivalent (up to shift) to the category $\mathcal{M}(H_C)$ of "staggered representations" considered in Section 1, and we may draw the following pictures:



When X contains more than one orbit, the structure of these categories is not so clear. This topic will be explored further in Section A.2.

4. Basic Properties

Perhaps the biggest technical difficulty in getting the theory of staggered sheaves off the ground is that the restriction functor j^*, where $j : U \hookrightarrow X$ is an open inclusion, does not have adjoints in the setting of coherent sheaves. (The direct image functor j_* gives quasicoherent sheaves in general, where the extension-by-zero functor $j_!$ takes values in the category of inverse limits of coherent sheaves.)

Theorem 4.1. *$({}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\le 0}, {}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\ge 0})$ is a nondegenerate, bounded t-structure on $\mathcal{D}^{\mathrm{b}}_G(X)$, so $\mathcal{M}(X)$ is an abelian category.*

Remarks on proof. This theorem cannot be proved using the method of "gluing of t-structures" [BBD, §1.4] that is typically used for perverse sheaves, because of the lack of adjoints mentioned above. Instead, we use the machinery of *baric structures*, discussed in Appendix A.

Theorem 4.2. *Every staggered sheaf has finite length, and there is a bijection*

$$\left\{ \begin{array}{c} \textit{simple objects} \\ \textit{in } \mathcal{M}(X) \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ (C,\mathcal{L}) \,\middle|\, \begin{array}{c} C \in \mathbb{O}(X), \textit{ and } \mathcal{L} \textit{ an irreducible} \\ \textit{vector bundle on } C \end{array} \right\}$$

The simple object corresponding to a pair $(C,\mathcal{L})$, denoted ${}^r\mathcal{IC}(C,\mathcal{L})$, or simply $\mathcal{IC}(C,\mathcal{L})$, is supported on $\overline{C}$, and its restriction to C is $\mathcal{L}[\operatorname{step}\mathcal{L} - r(C)]$.

Remarks on proof. Among all objects supported on $\overline{C}$ with the correct restriction to C, the objects $\mathcal{F}$ in the image of the functor $\mathcal{IC}(C,\cdot)$ are characterized by the property that for any $C' \subset \overline{C} \smallsetminus C$, we have

$$Li^*_{C'}\mathcal{F} \in {}^r\mathcal{D}^-_G(C')^{\le -1} \qquad \text{and} \qquad Ri^!_{C'}\mathcal{F} \in {}^r\mathcal{D}^+_G(C')^{\ge 1}.$$

This criterion can be used in examples to verify that some object is $\mathcal{IC}(C,\mathcal{L})$

Finally, we turn to duality. In the coherent setting, the appropriate analogue of the Poincaré–Verdier duality functor is the Serre–Grothendieck duality functor $\mathbb{D} : \mathcal{D}^{\mathrm{b}}_G(X) \to \mathcal{D}^{\mathrm{b}}_G(X)$, developed in [H] and generalized to the equivariant case in [B1]. To describe the behavior of $\mathcal{M}(X)$ under $\mathbb{D}$, we require some auxiliary definitions. Let us define a new function

$$\bar{r} : \mathbb{O}(X) \to \mathbb{Z},$$

known as the *dual perversity* to the given perversity $r : \mathbb{O}(X) \to \mathbb{Z}$, by the formula

$$\bar{r}(C) = \operatorname{step}\Omega^{\mathrm{top}}_C - \dim C - r(C). \tag{4.1}$$

We may then define categories

$${}^{\bar{r}}\mathcal{D}^{\mathrm{b}}_G(X)^{\le 0}, \quad {}^{\bar{r}}\mathcal{D}^{\mathrm{b}}_G(X)^{\ge 0}, \quad {}^{\bar{r}}\mathcal{M}(X)$$

in the same way was ${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\le 0}$, ${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\ge 0}$, and ${}^r\mathcal{M}(X)$, but using $\bar{r}$ in place of r (and using the same s-structure as before).

Theorem 4.3. *$\mathbb{D}({}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\le 0}) = {}^{\bar{r}}\mathcal{D}^{\mathrm{b}}_G(X)^{\ge 0}$ and $\mathbb{D}({}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\ge 0}) = {}^{\bar{r}}\mathcal{D}^{\mathrm{b}}_G(X)^{\le 0}$. In particular, $\mathbb{D}({}^r\mathcal{M}(X)) = {}^{\bar{r}}\mathcal{M}(X)$, and*

$$\mathbb{D}({}^r\mathcal{IC}(C,\mathcal{L})) \simeq {}^{\bar{r}}\mathcal{IC}(C,\mathcal{L}^\vee),$$

where $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \Omega^{\mathrm{top}}_C)$ is the Serre dual vector bundle to $\mathcal{L}$.

This statement becomes much cleaner, and closer to (2.3), when it happens that $r = \bar{r}$. Clearly, such a self-dual perversity exists if and only if the integer

$$\operatorname{step} \Omega_C^{\mathrm{top}} - \dim C,$$

known as the *staggered codimension* of C, is even for all $C \in \mathbb{O}(X)$. It is not known under what conditions this holds. (Similar considerations pertain to perverse sheaves as well, although in the algebraic setting, there is always a self-dual perversity—we silently chose it in Section 2—because the ℓ-adic cohomological dimension of every variety is even.)

5. Purity and Decomposition

We now describe a family of subcategories of $\mathcal{D}^{\mathrm{b}}_G(X)$ that are analogous to the weight categories $\mathrm{D}_{\leq w}$ and $\mathrm{D}_{\geq w}$ in ℓ-adic sheaf theory. Given an integer w, define

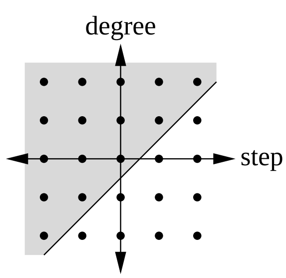
$${}^r\mathcal{D}^-_G(X)_{\sqsubseteq w} = \left\{ \mathcal{F} \;\middle|\; \begin{array}{c} h^k(Li_C^*\mathcal{F}) \in \mathcal{C}_G(C)_{\leq w + r(C) + \dim C + k} \\ \text{for all } C \in \mathbb{O}(X) \text{ and all } k \in \mathbb{Z} \end{array} \right\}.$$

As usual, when there is no risk of ambiguity, this category will simply be denoted $\mathcal{D}^-_G(X)_{\sqsubseteq w}$. Its definition clearly bears a great deal of resemblance to that of ${}^r\mathcal{D}^-_G(X)^{\leq w}$; the most essential difference is that the step constraint involves a "$+k$" rather than a "$-k$." Next, we define

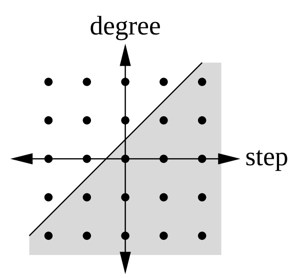
$${}^r\mathcal{D}^+_G(X)_{\sqsupseteq w} = \left\{ \mathcal{F} \;\middle|\; \begin{array}{c} h^k(Ri_C^!\mathcal{F}) \in \mathcal{C}_G(C)_{\geq w + r(C) + \dim C + k} \\ \text{for all } C \in \mathbb{O}(X) \text{ and all } k \in \mathbb{Z} \end{array} \right\}.$$

We also have bounded versions $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ and $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$. Over a single orbit, we may draw pictures of these categories resembling those for the ℓ-adic weight categories:

$\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$:



$\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$:



Of course, these categories are more complicated when X contains more than one orbit. Like the ℓ-adic weight categories, these categories enjoy a duality property, cf. (2.4) and Theorem 4.3:

Theorem 5.1. $\mathbb{D}({}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}) = {}^{\bar{r}}\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq -w}$ *and* $\mathbb{D}({}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}) = {}^{\bar{r}}\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq -w}$.

Our first result about them is similar to, but stronger than, the results (2.5) and (2.6) for ℓ-adic sheaves.

Theorem 5.2. *The collection of subcategories* $(\{\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}\}, \{\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}\})$ *forms a co-t-structure on* $\mathcal{D}^{\mathrm{b}}_G(X)$. *That is:*

1. $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} \subset \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w+1}$ *and* $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w} \supset \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$.
2. $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}[1] = \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w+1}$ *and* $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}[1] = \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$.
3. *If* $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ *and* $\mathcal{G} \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$, *then* $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0$.
4. *For any* $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$, *there is a distinguished triangle* $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ *with* $\mathcal{F}' \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ *and* $\mathcal{F}'' \in \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w+1}$.

Note that the inclusions $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} \subset \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}[1]$ and $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w} \supset \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}[1]$ are the opposite of what one would have in a t-structure. Also in contrast with t-structures, the distinguished triangle in part (4) above is not functorial in general.

Definition 5.3. An object $\mathcal{F} \in \mathcal{D}^{\mathrm{b}}_G(X)$ is *skew-pure* of *skew-degree* w if it belongs to $\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w} \cap \mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$.

Theorem 5.4 (Purity). *Every staggered sheaf $\mathcal{F}$ admits a canonical finite filtration*

$$\cdots \subset \mathcal{F}_{w-1} \subset \mathcal{F}_w \subset \mathcal{F}_{w+1} \subset \cdots$$

such that $\mathcal{F}_w/\mathcal{F}_{w+1}$ is skew-pure of skew degree w. In particular, every simple staggered sheaf is skew-pure.

Remarks on proof. The hard part of this is showing that a simple object is skew-pure, and the difficulty is that there is no general method to compute the restriction of $\mathcal{IC}(C,\mathcal{L})$ to $\overline{C} \smallsetminus C$. See Section A.3.

The skew degree of a simple staggered sheaf $\mathcal{F}$ is given by

$$\operatorname{sk\,deg} \mathcal{IC}(C,\mathcal{L}) = 2(\operatorname{step}\mathcal{L} - r(C)) - \dim C.$$

The next result follows immediately from properties of co-t-structures.

Proposition 5.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be simple staggered sheaves, with* $\operatorname{sk\,deg}\mathcal{F} = v$ *and* $\operatorname{sk\,deg}\mathcal{G} = w$. *Then* $\operatorname{Hom}^k(\mathcal{F},\mathcal{G}) = 0$ *for* $k > v - w$.

Here, $\operatorname{Hom}^k(\mathcal{F},\mathcal{G})$ is a synonym for $\operatorname{Hom}(\mathcal{F},\mathcal{G}[k])$. It can be identified with $\operatorname{Ext}^k(\mathcal{F},\mathcal{G})$ when $k \le 1$ (but not in general when $k > 1$). Thus, the proposition above includes an Ext^1-vanishing condition. A stronger statement about Ext^1-groups will appear in Corollary 6.5. See also Section 8.2.

Theorem 5.6 (Decomposition). *Every skew-pure object in $\mathcal{D}^{\mathrm{b}}_G(X)$ is a direct sum of shifts of simple staggered sheaves.*

6. Quasi-hereditary Abelian Categories

Let $\mathcal{A}$ be a finite-length $\Bbbk$-linear abelian category, and let S be the set of isomorphism classes of simple objects. Assume S is endowed with a fixed total order $\le$. For each $s \in S$, fix a representative object L_s. Assume also that $\operatorname{End}(L_s) \simeq \Bbbk$ for each $s \in S$.

Definition 6.1. An object M_s together with a surjective morphism $\phi_s : M_s \to L_s$ is called a *standard cover* if

1. Every simple subquotient of $\ker\phi_s$ is isomorphic to some L_t with $t < s$.
2. $\operatorname{Hom}(M_s, L_t) = \operatorname{Ext}^1(M_s, L_t) = 0$ for all $t < s$.

Dually, an object N_s together with an injective morphism $\psi_s : L_s \to N_s$ is called a *costandard hull* if

1. Every simple subquotient of $\operatorname{cok}\psi_s$ is isomorphic to some L_t with $t < s$.
2. $\operatorname{Hom}(L_t, N_s) = \operatorname{Ext}^1(L_t, N_s) = 0$ for all $t < s$.

Standard arguments show that standard covers and costandard hulls, when they exist, are unique up to canonical isomorphism. As noted at the end of Section 2, the motivating example of a standard object is a Verma modules in category $\mathcal{O}$ for a complex semisimple Lie algebra.

Note that a standard cover of L_s is a projective cover within the smaller category $\mathcal{A}_{\le s}$ generated by objects $\{L_t \mid t \le s\}$. Similarly, a costandard hull is an injective hull within $\mathcal{A}_{\le s}$.

Definition 6.2. The category $\mathcal{A}$ is said to be *quasi-hereditary* if

(6.1) Every simple object admits a standard cover and a costandard hull.

This definition, taken from [B1], is not the most common one. Many authors impose additional conditions, cf. Section 8.1.

Theorem 6.3. (1) *$\mathcal{M}(X)$ is quasi-hereditary.*
(2) *$\mathcal{M}(X)$ has enough projectives and enough injectives.*

Remarks on proof. The proof of (2.11) for perverse sheaves makes use of the functors $j_!$ and j_*, where $j : U \hookrightarrow X$ is an open inclusion. These are unavailable in the coherent setting. However, it turns out that the functor of abelian categories $j^* : \mathcal{M}(X) \to \mathcal{M}(U)$ has adjoints on both sides. These adjoints can be used in place of $j_!$ and j_*.

For a simple staggered sheaf $\mathcal{IC}(C, \mathcal{L})$, let $M(C, \mathcal{L})$ and $N(C, \mathcal{L})$ denote its standard cover and costandard hull, respectively.

Theorem 6.4. *Let $w = \operatorname{sk}\deg \mathcal{IC}(C, \mathcal{L})$.*
(1) *The kernel of $M(C, \mathcal{L}) \to \mathcal{IC}(C, \mathcal{L})$ is skew-pure of skew degree $w - 1$.*
(2) *The cokernel of $\mathcal{IC}(C, \mathcal{L}) \to N(C, \mathcal{L})$ is skew-pure of skew degree $w + 1$.*

Note that by the Decomposition Theorem, the kernel and cokernel mentioned in this theorem are both necessarily semisimple.

We conclude with two results whose statements do not involve standard or costandard objects, but whose proofs do.

Corollary 6.5. *If $\operatorname{Ext}^1(\mathcal{IC}(C, \mathcal{L}), \mathcal{IC}(C', \mathcal{L}')) \ne 0$, then*

$$\operatorname{sk}\deg \mathcal{IC}(C', \mathcal{L}') = \operatorname{sk}\deg \mathcal{IC}(C, \mathcal{L}) - 1.$$

Corollary 6.6. *For any orbit $C \in \mathbb{O}(X)$, the functor $\mathcal{IC}(C, \cdot) : \mathcal{M}(C) \to \mathcal{M}(X)$ embeds $\mathcal{M}(C)$ as a semisimple Serre subcategory of $\mathcal{M}(X)$.*

The semisimplicity of $\mathcal{M}(C)$ itself was essentially stated in Section 1. The additional content of this corollary is that even in the larger category $\mathcal{M}(X)$, there are no nontrivial extensions about simple objects supported on the same orbit closure.

7. An Example

Let G denote the usual Borel subgroup of $PGL(2, \Bbbk)$:

$$G = \left\{ \begin{bmatrix} * & * \\ & * \end{bmatrix} \right\} \Big/ \left\{ \pm \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right\}$$

Let $X = \mathbb{P}^1$, endowed with the obvious action of G. (Note that X is the flag variety for $PGL(2, \Bbbk)$. G has two orbits on $\mathbb{P}^1$: a one-point orbit, denoted Z, and its complement, an open orbit isomorphic to $\mathbb{A}^1$ and denoted U. The isotropy groups are $H_Z = G$ and $H_U \simeq \mathbb{G}_{\mathrm{m}}$. Irreducible representations of both these groups are in bijection with $\mathbb{Z}$. For $n \in \mathbb{Z}$, let $V_n \in \mathcal{C}_G(Z)$ and $\mathcal{L}_n \in \mathcal{C}_G(U)$ denote the corresponding vector bundles on orbits.

Note that the torus $H_U \subset G$ acts linearly on U with weight 1. It can be deduced from this that $N^*(Z) \simeq \mathfrak{u}_Z \simeq V_1$. It also follows that for any two integers $n, m \in \mathbb{Z}$, there exists a line bundle $\mathcal{F}(n,m) \in \mathcal{C}_G(X)$ such that $\mathcal{F}(n,m)|_U \simeq \mathcal{L}_n$ and $i_Z^*\mathcal{F}(n,m) \simeq V_m$. The canonical bundle is $\Omega_X^{\mathrm{top}} = \mathcal{F}(-1,1)$, and Serre–Grothendieck duality is given by

$$\mathbb{D}(\mathcal{F}(n,m)) \simeq \mathcal{F}(-1-n, 1-m)[1].$$

We impose an s-structure on X by requiring $\operatorname{step} \mathcal{L}_n = n$ and $\operatorname{step} V_n = -n$, and we give it the self-dual perversity $r : U \mapsto -1$, $Z \mapsto 0$. Then, the simple staggered sheaves are given by

$$\mathcal{IC}(U, \mathcal{L}_n) \simeq \mathcal{F}(n, -n)[n+1] \qquad \text{and} \qquad \mathcal{IC}(Z, V_n) \simeq V_n[-n].$$

The only nontrivial extensions among simple objects are the following:

$$0 \to \mathcal{IC}(Z, V_{-n}) \to \mathcal{F}(n, -n+1)[n+1] \to \mathcal{IC}(U, \mathcal{L}_n) \to 0$$
$$0 \to \mathcal{IC}(U, \mathcal{L}_n) \to \mathcal{F}(n, -n-1)[n+1] \to \mathcal{IC}(Z, V_{-n-1}) \to 0$$

The middle term of each of those is both projective and injective. In fact, we have

$$M(U, \mathcal{L}_n) \simeq P(U, \mathcal{L}_n) \simeq \mathcal{F}(n, -n+1)[n+1] \simeq I(Z, V_{-n})$$
$$N(U, \mathcal{L}_n) \simeq I(U, \mathcal{L}_n) \simeq \mathcal{F}(n, -n-1)[n+1] \simeq P(Z, V_{-n-1})$$

Since Z is closed, the simple objects $\mathcal{IC}(Z, V_n)$ are also standard and costandard.

The structure of the category $\mathcal{M}(X)$ can be summarized in the following diagram, in which an arrow $\mathcal{F} \to \mathcal{G}$ means that $\mathrm{Ext}^1(\mathcal{G}, \mathcal{F}) \neq 0$.

$$\begin{tikzcd}
\text{sk deg}: & & -2 & -1 & 0 & 1 & 2 & 3 & \\
U: & \cdots & & \mathcal{IC}(U,-1) \arrow[dl] & & \mathcal{IC}(U,0) \arrow[dl] & & \mathcal{IC}(U,1) \arrow[dl] & \cdots \\
Z: & & \mathcal{IC}(Z,1) & & \mathcal{IC}(Z,0) \arrow[ul] & & \mathcal{IC}(Z,-1) \arrow[ul] & &
\end{tikzcd}$$

It can also be checked by direct calculation that derived tensor products

$$\mathcal{IC}(Z, V_n) \overset{L}{\otimes} \mathcal{IC}(Z, V_m)$$

are skew-pure, of skew-degree $-2n - 2m$. The Decomposition Theorem tells us these must be semisimple; we have

$$\mathcal{IC}(Z, V_n) \overset{L}{\otimes} \mathcal{IC}(Z, V_m) \simeq \mathcal{IC}(Z, V_{n+m}) \oplus \mathcal{IC}(Z, V_{n+m+1})[2].$$

Further explicit examples appear in [A1, §11], [AT2, §12], and [A2, §9]. The example in [AT2, §12] gives another illustration of the Decomposition Theorem. The example in [A2, §9] involves a nonsmooth variety.

8. Further Questions

8.1. **Reciprocity formulas.** Let $\mathcal{A}$ and S be as in Section 6. The notion of quasi-hereditarity often includes the following additional assumptions:

(8.1) The set S of isomorphism classes of simple objects is finite.

(8.2) For any standard object M_s and costandard object N_t, $\mathrm{Ext}^2(M_s, N_t) = 0$.

(Both of these fail for $\mathcal{M}(X)$ in general.) These extra conditions always imply the existence of enough projectives and injectives [BGS, Theorem 3.2.1] (see also [CPS]). Moreover, every indecomposable projective admits a *standard filtration*, i.e., a filtration whose subquotients are standard objects. The multiplicities in a standard filtration obey the celebrated *Brauer–Humphreys reciprocity* formula:

$$(P_s : M_t) = [N_t : L_s]. \tag{8.3}$$

Perhaps the most famous instance of this is the one known as *BGG reciprocity* in category $\mathcal{O}$ for a complex semisimple Lie algebra.

The known examples of staggered sheaves include one in which projectives do have standard filtrations (see [A2, §9]) and others (including the one in Section 7) in which they do not. It would be nice to find a characterization of cases in which this happens, and therefore in which the reciprocity formula (8.3) holds.

8.2. **Higher Ext-groups.** It can be seen in the smallest examples, including the one in Section 7, that the bounded derived category of $\mathcal{M}(X)$ need not be equivalent to $\mathcal{D}^{\mathrm{b}}_G(X)$, so care must be taken in passing from higher Hom-groups in $\mathcal{D}^{\mathrm{b}}_G(X)$ to higher Ext-groups for the abelian category $\mathcal{M}(X)$. If $\mathcal{F}$ and $\mathcal{G}$ are simple staggered sheaves with $\operatorname{sk}\deg\mathcal{F} \le \operatorname{sk}\deg\mathcal{G}$, Proposition 5.5 tells us that $\operatorname{Hom}^k(\mathcal{F},\mathcal{G}) = 0$ for all $k > 0$, and then it follows from [BBD, Remarque 3.1.17] that

$$\operatorname{Ext}^k(\mathcal{F},\mathcal{G}) = 0 \text{ for all } k > 0 \text{ if } \operatorname{sk}\deg\mathcal{F} \le \operatorname{sk}\deg\mathcal{G}. \tag{8.4}$$

It is natural to ask what can be said about higher Ext-groups when $\operatorname{sk}\deg\mathcal{F} > \operatorname{sk}\deg\mathcal{G}$. It follows from (8.4) that $\mathcal{M}(X)$ is a *mixed category* in the sense of [BGS, §4]. In keeping with the themes of that paper, one may ask whether $\mathcal{M}(X)$ is, in fact, a *Koszul category.* In other words, is it true that

$$\operatorname{Ext}^k(\mathcal{F},\mathcal{G}) = 0 \qquad \text{if } \operatorname{sk}\deg\mathcal{F} - \operatorname{sk}\deg\mathcal{G} \ne k?$$

Corollary 6.5 says this holds for $k = 1$; it seems reasonable to conjecture that it holds in general.

8.3. **Cohomological dimension of $\mathcal{M}(X)$.** It is well-known that the cohomological dimension of the category $\mathcal{C}_G(X)$ is $\dim X$ if X is smooth and infinite otherwise. For staggered sheaves, the situation is quite different: in the example considered in Section 7, $\mathcal{M}(X)$ has infinite cohomological dimension, but in every known example on an affine variety, including a nonsmooth one [A2, §9], $\mathcal{M}(X)$ has finite cohomological dimension. It would be interesting to find necessary and sufficient conditions for $\mathcal{M}(X)$ to have finite cohomological dimension, and to compute its dimension in those cases.

Appendix A. Baric Structures

A.1. **Overview.** A vital role in the proofs of all the main results on staggered sheaves is played by the notion of a *baric structure*, introduced by the author and D. Treumann in [AT1]. The motivating example of a baric structure appeared earlier, in work of S. Morel, in the ℓ-adic setting [M]. The definition is given as part of Theorem A.2 below.

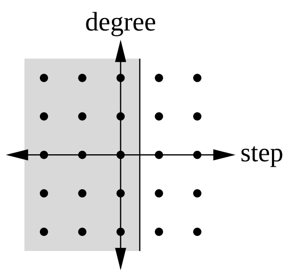
The paper [AT1] includes an extensive study of so-called *HLR baric structures.* These baric structures are *hereditary* (well-behaved on closed subschemes), *local* (well-behaved on open subschemes), and *rigid* (well-behaved on nilpotent thickenings). In additional to the general theory, that paper includes a construction of a specific family of HLR baric structures, which we now describe.

Let $q : \mathbb{O}(X) \to \mathbb{Z}$ be a function, and define a collection of full subcategories of $\mathcal{D}^-_G(X)$ and $\mathcal{D}^+_G(X)$, respectively, as follows:

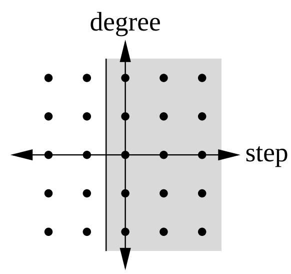
$$
\begin{aligned}
{}_q\mathcal{D}^-_G(X)_{\le w} &= \{\mathcal{F} \mid h^k(Li^*_C\mathcal{F}) \in \mathcal{C}_G(C)_{\le q(C)+w} \text{ for all } C \in \mathbb{O}(X) \text{ and all } k \in \mathbb{Z}\},\\
{}_q\mathcal{D}^+_G(X)_{\ge w} &= \{\mathcal{F} \mid h^k(Ri^!_C\mathcal{F}) \in \mathcal{C}_G(C)_{\ge q(C)+w} \text{ for all } C \in \mathbb{O}(X) \text{ and all } k \in \mathbb{Z}\}.
\end{aligned}
$$

As usual, let ${}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w}$ and ${}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w}$ denote the bounded versions of these categories. These categories are stable under shift. Over a single orbit, they look like this:

${}_q\mathcal{D}^\mathrm{b}_G(C)_{\le w}$:



${}_q\mathcal{D}^\mathrm{b}_G(C)_{\ge w}$:



These categories exhibit a duality property similar to those in Sections 4 and 5.

Theorem A.1. $\mathbb{D}({}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w}) = {}_{\hat{q}}\mathcal{D}^\mathrm{b}_G(X)_{\ge -w}$ *and* $\mathbb{D}({}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w}) = {}_{\hat{q}}\mathcal{D}^\mathrm{b}_G(X)_{\le -w}$, *where* $\hat{q}(C) = \operatorname{step}\Omega^{\mathrm{top}}_C - q(C)$.

Theorem A.2. *The collection of subcategories* $(\{{}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w}\}, \{{}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w}\})$ *forms a baric structure on* $\mathcal{D}^\mathrm{b}_G(X)$. *That is:*

1. ${}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w} \subset {}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w+1}$ *and* ${}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w} \supset {}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w+1}$.
2. *If* $\mathcal{F} \in {}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w}$ *and* $\mathcal{G} \in {}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w+1}$, *then* $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0$.
3. *For any* $\mathcal{F} \in \mathcal{D}^\mathrm{b}_G(X)$, *there is a distinguished triangle* $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ *with* $\mathcal{F}' \in {}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w}$ *and* $\mathcal{F}'' \in {}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w+1}$.

In a baric structure, as in a t-structure (but unlike in a co-t-structure), the distinguished triangle in the last property above is functorial. Specifically, there are *baric truncation functors*

$$
{}_q\beta_{\le w} : \mathcal{D}^\mathrm{b}_G(X) \to {}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w} \qquad \text{and} \qquad {}_q\beta_{\ge w} : \mathcal{D}^\mathrm{b}_G(X) \to {}_q\mathcal{D}^\mathrm{b}_G(X)_{\ge w}.
$$

A.2. **Staggering operation.** The baric structure above, like the t-structure of Section 3.2 and the co-t-structure of Section 5, is defined by "orbitwise" conditions, but all three of these admit an alternate description by "cohomologywise" conditions. Define a subcategory of $\mathcal{C}_G(X)$ as follows:

$$
{}_q\mathcal{C}_G(X)_{\le w} = \{\mathcal{F} \mid i^*_C\mathcal{F} \in \mathcal{C}_G(C)_{\le q(C)+w} \text{ for all } C \in \mathbb{O}(X)\}.
$$

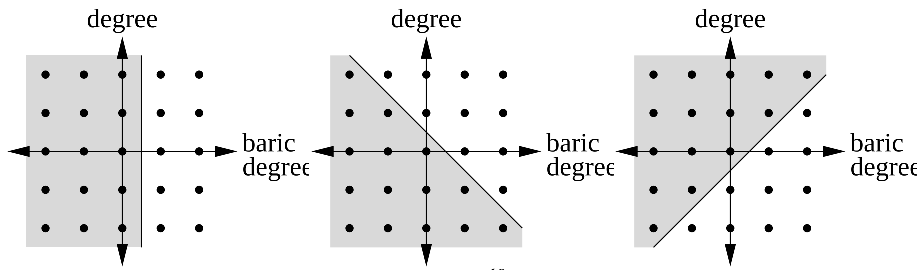
The descriptions appearing in the proposition below were actually taken as definitions in [AT1, AT2] (and the orbitwise descriptions required proof).

Proposition A.3. *Let* $\mathcal{F} \in \mathcal{D}^\mathrm{b}_G(X)$.

1. $\mathcal{F} \in {}_q\mathcal{D}^\mathrm{b}_G(X)_{\le w}$ *if and only if* $h^k(\mathcal{F}) \in {}_q\mathcal{C}_G(X)_{\le w}$ *for all* $k \in \mathbb{Z}$.
2. $\mathcal{F} \in {}^r\mathcal{D}^\mathrm{b}_G(X)^{\le w}$ *if and only if* $h^k(\mathcal{F}) \in {}_r\mathcal{C}_G(X)_{\le w-k}$ *for all* $k \in \mathbb{Z}$.

(3) *$\mathcal{F} \in {}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ if and only if $h^k(\mathcal{F}) \in {}_{\llcorner r \lrcorner}\mathcal{C}_G(X)_{\leq w+k}$ for all $k \in \mathbb{Z}$, where*
$$\llcorner r \lrcorner(C) = r(C) + \dim C.$$

This really is a one-sided statement: in general, membership in ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geq w}$, ${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\geq 0}$, or ${}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$ cannot directly be tested for on cohomology sheaves. For ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leq w}$, ${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}$, and ${}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$, however, it is now reasonable to draw pictures even in the many-orbit case, with the horizontal axis now indicating membership in a ${}_q\mathcal{C}_G(X)_{\leq w}$ rather than step.



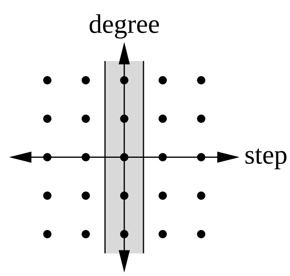
${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leq w}$ ${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}$ ${}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$

The advantage of the descriptions in Proposition A.3 is that the relationship between ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leq w}$ and ${}^r\mathcal{D}^{\mathrm{b}}_G(X)^{\leq 0}$ can now be described in language that makes sense in any triangulated category.

The *staggering operation* is a procedure that takes "compatible" baric and t-structures on a triangulated category, and produces from them a new t-structure. The latter is defined by cohomology conditions. Theorem 4.1 is proved by invoking this general procedure.

A.3. **Baric purity.** The full subcategory ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]} = {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leq w} \cap {}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\geq w}$, shown below for a single orbit, is a triangulated category in its own right, and one may ask whether it admits any interesting t-structures.

${}_q\mathcal{D}^{\mathrm{b}}_G(C)_{[w]}$:



Two t-structures on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ (known as the *purified* and the *pure-perverse* t-structures) are defined and studied in [AT2]. Neither of these is induced by one on $\mathcal{D}^{\mathrm{b}}_G(X)$, but they nevertheless turn out to be useful for studying particular objects. Specifically, the key to proving the Purity Theorem, Theorem 5.4, is showing that a simple staggered sheaf $\mathcal{IC}(C, \mathcal{L})$ also lies in the heart of the pure-perverse t-structure on ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{[w]}$ for a suitable choice of q and w.

Along the way, it is shown in [AT2] that when the perversity functions obey certain inequalities, baric versions of the Purity and Decomposition Theorems hold.

Theorem A.4. *Under certain assumptions on r and q, the following hold.*

(1) *Every staggered sheaf admits a canonical finite filtration with baric-pure subquotients. In particular, every simple staggered sheaf is baric-pure.*

(2) *Every baric-pure object in $\mathcal{D}^{\mathrm{b}}_G(X)$ is a direct sum of shifts of simple staggered sheaves.*

A.4. **Nonreduced schemes.** Although X was assumed to be a (reduced) variety throughout in the present note, it is in fact essential to allow nonreduced schemes in general, since the scheme-theoretic support of a coherent sheaf on a variety may be nonreduced. The "rigidity" property of HLR baric structures plays an important role here: for nonreduced X, membership in (for instance) ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\le w}$ can be tested for after pulling back to ${}_q\mathcal{D}^{\mathrm{b}}_G(X_{\mathrm{red}})_{\le w}$, where X_{red} is the associated reduced scheme. (This was not known when [A1] was written, so definitions in that paper are typically quantified over all closed subschemes, rather than just all reduced closed subschemes.)

However, the following result from [A2] depends on the quasi-hereditary property and does not seem to follow directly from rigidity.

Theorem A.5. *Let X be a nonreduced scheme, and let $t : X_{\mathrm{red}} \to X$ be the inclusion of the associated reduced scheme. Then $t_* : \mathcal{M}(X_{\mathrm{red}}) \to \mathcal{M}(X)$ is an equivalence of categories.*

Appendix B. Changes in Conventions and Notation

Staggered sheaves were introduced in [A1] in a much more general setting than we have considered here: G and X were schemes of finite type over a noetherian base scheme admitting a dualizing complex, with no assumption on orbits. With hindsight, this setting appears to have been too general: the key results really do require X to be a variety with finitely many orbits. The latter setting permits great simplifications of the basic definitions, and thus the definitions in the present note may seem to bear little resemblance to their counterparts in [A1] and subsequent papers. In this appendix, we briefly indicate how to connect the notions in this note to those in the various papers.

B.1. **s-structures.** In [A1], an *s-structure* on X is a collection of full subcategories $(\{\mathcal{C}_G(X)_{\le w}\}, \{\mathcal{C}_G(X)_{\ge w}\})_{w\in\mathbb{Z}}$ of $\mathcal{C}_G(X)$, satisfying a rather lengthy list of axioms. However, on a variety with finitely many orbits, the "gluing theorems" [A1, Theorem 5.3] and [AS, Theorem 2.1] allow one to reduce the task of specifying an s-structure to specifying one on each orbit, and by [AT2, §7], the latter is equivalent to giving a certain central cocharacter of the isotropy group.

The definition in (3.4) actually corresponds to what have been called "recessed, split s-structures" in [AT1, AT2, A2]. The main theorems all require recessed, split s-structures, so in the present note, that terminology has been dropped, and those conditions incorporated into the definition of "s-structure."

B.2. **Perversities.** In [A1], a perversity function was required to obey certain inequalities, known as the "monotone" and "comonotone" conditions, in order for Theorem 4.1 to hold, and Theorem 4.2 was proved only for perversities obeying even stronger inequalities ("strictly monotone and comonotone"). These restrictions were removed in [AT1] by entirely new proofs using baric structures. (These new proofs assume "recessed" s-structures, however.) Thus, a perversity function is now an arbitrary function $\mathbb{O}(X) \to \mathbb{Z}$.

B.3. **Duality and codimension.** In this note, the Serre–Grothendieck duality functor $\mathbb{D}$ has been defined in terms of "the" dualizing complex ω_X, using the assumption (3.1) to guarantee its existence in positive characteristic. Let us now take *dualizing complex* to mean any object $\tilde{\omega}_X$ with the property that the functor $\mathrm{R}\mathcal{H}om(\cdot, \tilde{\omega}_X)$ is an antiautoequivalence of $\mathcal{D}^{\mathrm{b}}_G(X)$. Such dualizing complexes exist in very great generality (see [H, §V.10] and [B1, Proposition 1]), without any assumption like (3.1). In the papers [A1, AT1, AT2], the set-up included the *choice* of a dualizing complex. No results on staggered sheaves depend in a substantive way on this choice, but various formulas such as (4.1) must be modified.

Specifically, for any orbit C, the object $i^!_C\tilde{\omega}_X$ must be a shift of a line bundle; say $i^!_C\tilde{\omega} \simeq \mathcal{L}_C[n_C]$. The *altitude* of C, denoted $\operatorname{alt} C$, is defined to be $\operatorname{step}\mathcal{L}_C$. We also put $\operatorname{cod} C = -n_C$. The assumptions of the present note give $\operatorname{alt} C = \operatorname{step}\Omega^{\mathrm{top}}_C$ and $\operatorname{cod} C = -\dim C$. The latter assumption was also made in [A2].

B.4. **Baric and co-t-structures.** The definition of ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\leq w}$ used in Appendix A follows [AT1], but the definition used in [AT2] contains an extra factor of 2, for reasons explained therein. Co-t-structures were introduced in [AT2] in a way that depended on an additional function $q : \mathbb{O}(X) \to \mathbb{Z}$, and the associated categories were denoted ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ and ${}_q\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$. These coincide with the ${}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsubseteq w}$ and ${}^r\mathcal{D}^{\mathrm{b}}_G(X)_{\sqsupseteq w}$ of the present note when $q = \llcorner r \lrcorner$, cf. Proposition A.3.

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