LOCAL SYSTEMS ON NILPOTENT ORBITS AND WEIGHTED DYNKIN DIAGRAMS

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1. Introduction

Let G be a reductive algebraic group over the complex numbers, B a Borel subgroup of G, and T a maximal torus of B. We denote by $\Lambda = \Lambda(G)$ the weight lattice of G with respect to T, and by $\Lambda_+ = \Lambda_+(G)$ the set of dominant weights with respect to the positive roots defined by B. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}$.

Now, let $e \in \mathcal{N}$ be a nilpotent element, and let $\mathcal{O}_e$ be the orbit of e in $\mathfrak{g}$ under the adjoint action of G. We write G^e for the centralizer of e in G. Let $\mathcal{N}_o$ denote the set of nilpotent orbits in $\mathfrak{g}$, and $\mathcal{N}_{o,r}$ the set of G-conjugacy classes of pairs

$$\{(e, \tau) \mid e \in \mathcal{N} \text{ and } \tau \text{ an irreducible rational representation of } G^e\}.$$

Lusztig [9] conjectured the existence of a bijection $\mathcal{N}_{o,r} \leftrightarrow \Lambda_+$ using his work on cells in affine Weyl groups. From the point of view of Harish-Chandra modules, Vogan also conjectured a bijection between $\mathcal{N}_{o,r}$ and Λ_+. Such a bijection has been established by Bezrukavnikov in two preprints (the bijections in each preprint are conjecturally the same) [2], [3]. Bezrukavnikov's second bijection is closely related to Ostrik's conjectural description of the bijection [12] (see also [4]). In the case of $G = GL(n, \mathbb{C})$, the first author [1] described an explicit combinatorial bijection between $\mathcal{N}_{o,r}$ and Λ_+ from the Harish-Chandra module perspective. At present, it is not known how all of these bijections are related (Bezrukavnikov's two candidates; Ostrik's conjectural candidate; and the first author's candidate in type A). In this paper, we work in the context of [1], which we now review.

Let $K_G(\mathcal{N})$ denote the Grothendieck group of G-equivariant coherent sheaves on $\mathcal{N}$. On the one hand, $K_G(\mathcal{N})$ has a natural basis indexed by elements of Λ_+, denoted $AJ(\lambda)$ in [12] (but unlike in [12], we are not utilizing the $\mathbb{C}^*$ action on $\mathcal{N}$ here). The algebraic global sections of $AJ(\lambda)$ are isomorphic as a G-module to $\operatorname{Ind}_T^G \lambda$. Thus, the space of global sections of any element $\mathcal{F} \in K_G(\mathcal{N})$ is given as a G-module by a unique expression of the form

$$\Gamma(\mathcal{N}, \mathcal{F}) = \sum_{\lambda \in \Lambda_+} m_\lambda \operatorname{Ind}_T^G \lambda, \tag{1}$$

where the $m_\lambda \in \mathbb{Z}$, and $m_\lambda \neq 0$ for only finitely many λ. (This fact was communicated to us by Vogan.)

Let us now fix a pair $(e, \tau) \in \mathcal{N}_{o,r}$. We want to consider all elements $\mathcal{F} \in K_G(\mathcal{N})$ whose support is contained in $\overline{\mathcal{O}}_e$, and whose restriction to $\mathcal{O}_e$ is the vector bundle arising from τ.

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For each such $\mathcal{F}$, there is at least one λ of maximal length occurring in the expression (1) (where we have fixed a W-invariant positive-definite symmetric bilinear form on the real span of Λ, so that we can speak of the length of a weight of G). Define $\gamma : \mathcal{N}_{o,r} \to \Lambda_+$ by

$$\gamma(e,\tau) = \text{the } \textit{smallest} \text{ such largest } \lambda, \text{ over all possible choices of } \mathcal{F}.$$

The following conjecture was made in [1]; moreover, it was proved in the case of $G = GL(n,\mathbb{C})$ in *op. cit.*

Conjecture 1.1 ([1])**.** *The map γ is well-defined and is a bijection. Moreover, there is a basis $\{M(e,\tau)\}$ for $K_G(\mathcal{N})$, indexed by $\mathcal{N}_{o,r}$, such that*

(1) *the support of $M(e,\tau)$ is $\overline{\mathcal{O}}_e$,*
(2) *the restriction of $M(e,\tau)$ to $\mathcal{O}_e$ is the locally free sheaf arising from τ, and*
(3) *there is an upper-triangular relationship between this basis and the one indexed by Λ_+:*

$$\Gamma(\mathcal{N}, M(e,\tau)) = \pm \operatorname{Ind}_T^G \gamma(e,\tau) + \sum_{\|\mu\|^2<\|\tau\|^2} m_\mu \operatorname{Ind}_T^G \mu.$$

In this paper, we study the problem of computing γ when τ gives rise to a local system on $\mathcal{O}_e$ (for semisimple groups, this means that τ is trivial on the identity component of G^e). Let us denote by Λ_+^{loc} the image of γ when τ corresponds to a local system.

Here is an outline of the paper. In section 2, we compute Λ_+^{loc} explicitly for $G = GL(n)$. In section 3, we state a precise conjecture, for general G, that the Dynkin weights for the Langlands dual group of G are a subset of Λ_+^{loc}. In section 4, we show how to associate to $\gamma(e,\tau)$ a sub-bundle of the cotangent bundle of G/B. Then for $GL(n)$, we are able to prove that when e is in a fixed nilpotent orbit, the cohomology of the sub-bundles (with coefficients in the structure sheaf) are independent of the local system τ.

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2. Computing γ for a local system in $GL(n)$

For a partition $\mathbf{d}$ of n, let $\mathcal{O}_{\mathbf{d}}$ denote the nilpotent orbit in $\mathfrak{gl}(n)$ indexed by $\mathbf{d}$. If $\mathbf{d} = [k_1^{a_1}, k_2^{a_2}, \ldots, k_l^{a_l}]$, then we know that $\pi_1(\mathcal{O}_{\mathbf{d}}) \simeq \mathbb{Z}/c\mathbb{Z}$, where c is the greatest common divisor of the k_i's [5]. Let $G^{\mathbf{d}}$ denote the isotropy group of an element in this orbit, and $G^{\mathbf{d}}_{\text{red}}$ the reductive part thereof. Following the notation of [5], we have

$$G^{\mathbf{d}}_{\text{red}} \simeq GL(a_1)^{k_1}_{\Delta} \times \cdots \times GL(a_l)^{k_l}_{\Delta}.$$

Here, H^m_{Δ} denotes the diagonal copy of H in the product of m copies of itself. Now, in the case of simply connected semisimple groups, we identify local systems on an orbit with representations of the component group of the centralizer of an element in the orbit. In the setting of $GL(n)$, we will first produce a list of representations of $G^{\mathbf{d}}_{\text{red}}$, and then we will examine their restrictions to $SL(n)$ to determine the correspondence with local systems.

Let $A_1 \in GL(a_1)$, $A_2 \in GL(a_2)$, etc. The set of matrices $(A_1, \ldots, A_l)$ determines an element $\tilde{A}$ of

$$GL(a_1)^{k_1}_{\Delta} \times \cdots \times GL(a_l)^{k_l}_{\Delta};$$

moreover, we have

$$\det \tilde{A} = (\det A_1)^{k_1} \cdots (\det A_l)^{k_l}.$$

Since every k_i is divisible by c, we can write this as

$$\det \tilde{A} = \left((\det A_1)^{k_1/c} \cdots (\det A_l)^{k_l/c}\right)^c.$$

Let "$\det^{1/c}$" denote the character $G^{\mathbf{d}}_{\text{red}} \to \mathbb{C}^\times$ given by

$$(\det A_1)^{k_1/c} \cdots (\det A_l)^{k_l/c},$$

and for any $p \in \mathbb{Z}$, let $\det^{p/c}$ denote the p-th power of this map.

In $SL(n)$, the isotropy group $SL(n)^{\mathbf{d}}$ is the subgroup of $G^{\mathbf{d}}$ consisting of matrices of determinant 1. The function $\det^{1/c}$ takes only finitely many values when restricted to this subgroup, namely, the c-th roots of unity. Indeed, the component group of $SL(n)^{\mathbf{d}}$ is just $\mathbb{Z}/c\mathbb{Z}$, and the irreducible representations of $\mathbb{Z}/c\mathbb{Z}$ are just powers of the function $\det^{1/c}$. We have, then, that the irreducible representations of $\pi_1(\mathcal{O}_{\mathbf{d}})$ come from

$$\det^{0/c}, \det^{1/c}, \ldots, \det^{(c-1)/c}.$$

Now we need to compute $\gamma(\mathcal{O}_{\mathbf{d}}, \det^{p/c})$ for $0 \le p \le c-1$. Suppose that the dual partition of $\mathbf{d}$ is $\mathbf{d}^{\mathrm{t}} = [j_1^{b_1}, \ldots, j_s^{b_s}]$. Following the notation of [1], we let $L_{\mathbf{d}}$ denote the Levi subgroup

$$L_{\mathbf{d}} = GL(j_1)^{b_1} \times \cdots \times GL(j_s)^{b_s},$$

and we let $P_{\mathbf{d}}$ be the standard (block-upper-triangular) parabolic subgroup whose Levi factor is $L_{\mathbf{d}}$. Recall that every nilpotent orbit in $GL(n)$ is Richardson; in particular, $\mathcal{O}_{\mathbf{d}}$ is Richardson for $P_{\mathbf{d}}$. We fix a choice of $e \in \mathcal{O}_{\mathbf{d}}$ such that $G^{\mathbf{d}}_{\text{red}} \subset L_{\mathbf{d}}$ and $G^{\mathbf{d}} \subset P_{\mathbf{d}}$. Explicitly, the embedding of $G^{\mathbf{d}}$ in $P_{\mathbf{d}}$ is such that the factor $GL(a_i)^{k_i}_{\Delta}$ sits diagonally across the product of the first k_i factors of $L_{\mathbf{d}}$. Let $\rho_{\mathbf{d}}$ denote the half-sum of the positive roots of $L_{\mathbf{d}}$.

We make the observation that any irreducible representation of $G^{\mathbf{d}}$ must be trivial on its unipotent radical, hence is completely determined by its restriction to $G^{\mathbf{d}}_{\text{red}}$. Therefore, it makes sense to refer to irreducible representations of $G^{\mathbf{d}}$ by their highest weights. We do this at various points below.

In [1], the computational device of "weight diagrams" was introduced and employed to carry out the computation of γ. For our present purposes, however, we are only concerned with a small collection of representations for each orbit, and we do not need to use the cumbersome weight diagrams. Instead, we make use of the following auxiliary result.

Proposition 2.1 ([1], Claim 2.3.1)**.** *Let (π_λ, V_λ) denote the irreducible $L_{\mathbf{d}}$-representation of highest weight λ, regarded as a $P_{\mathbf{d}}$-representation by letting the unipotent part of $P_{\mathbf{d}}$ act trivially, and let (τ, W_τ) be an irreducible representation of $G^{\mathbf{d}}$. Suppose that τ occurs as a summand in the restriction of π_λ to $G^{\mathbf{d}}$, and, moreover, that*

$$\|\lambda + 2\rho_{\mathbf{d}}\|^2$$

is minimal among all irreducible $P_{\mathbf{d}}$-representations whose restriction to $G^{\mathbf{d}}$ contains τ as a summand. Then $\gamma(\mathcal{O}_{\mathbf{d}}, \tau) = \lambda + 2\rho_{\mathbf{d}}$, made dominant for $GL(n)$.

The next proposition describes $\gamma(\mathcal{O}_{\mathbf{d}}, \det^{p/c})$ explicitly in terms of the standard basis. The remainder of the section will be devoted to establishing this proposition. In Section 4, we shall use this explicit description to prove the main result.

Let $\omega_p = \gamma(\mathcal{O}_{\mathbf{d}}, \det^{p/c})$. Writing ω_0 in the standard basis, let "block B_a" refer to the collection of coordinate positions which contain the entry a in ω_0, for $a \in \mathbb{Z}$. Let μ_k be the multiplicity of k in $\mathbf{d}^{\mathrm{t}}$. (Thus, if there is some j_i such that $j_i = k$, then $\mu_k = b_i$; otherwise, $\mu_k = 0$.) Note that every b_i and every μ_k must be a multiple of c.

Proposition 2.2. *The length of block* B_a *is* $\sum_{k\geq 0} \mu_{a+2k+1}$ *when* $a \geq 0$ *and the length of* B_a *and* B_{-a} *are the same. Moreover, if we write the length of block* B_a *as* $m_a c$*, then* ω_p *is obtained by replacing the first* $m_a p$ *entries in block* B_a *with* $a+1$*.*

Example 2.3. Consider the orbit labeled by $\mathbf{d} = [6, 3, 3]$ in $GL(12)$. Then $c = 3$, $\mathbf{d}^{\mathrm{t}} = [3, 3, 3, 1, 1, 1]$, and $\mu_3 = 3$ and $\mu_1 = 3$.

We illustrate the preceding proposition by listing here all the ω_p. The first sentence of the proposition describes ω_0 by giving the lengths of blocks, and the second sentence tells us how to obtain the other ω_p's by modifying ω_0.

$$\begin{aligned}
\omega_0 &= (\overbrace{2, 2, 2}^{\mathrm{B}_2}, \overbrace{0, 0, 0, 0, 0, 0}^{\mathrm{B}_0}, \overbrace{-2, -2, -2}^{\mathrm{B}_{-2}}) \\
\omega_1 &= (3, 2, 2, 1, 1, 0, 0, 0, 0, -1, -2, -2) \\
\omega_2 &= (3, 3, 2, 1, 1, 1, 1, 0, 0, -1, -1, -2)
\end{aligned}$$

Proof. Let λ_p be the appropriate dominant weight of $L_{\mathbf{d}}$ as described in Proposition 2.1, such that ω_p is just $\lambda_p + 2\rho_{\mathbf{d}}$, made dominant for $GL(n)$. In what follows, we shall be careless about saying "made dominant" every time; the reader should fill in those words wherever appropriate.

Let us begin with the trivial representation of $\pi_1(\mathcal{O}_{\mathbf{d}})$, namely $\det^{0/c}$. Whatever λ_0 is, it must be dominant for $L_{\mathbf{d}}$, so that $\langle \lambda_0, 2\rho_{\mathbf{d}} \rangle \geq 0$. Therefore

$$\|\lambda_0 + 2\rho_{\mathbf{d}}\|^2 = \|\lambda_0\|^2 + \|2\rho_{\mathbf{d}}\|^2 + 2\langle \lambda_0, 2\rho_d d \rangle \geq \|2\rho_{\mathbf{d}}\|^2.$$

Now 0 is a weight of $L_{\mathbf{d}}$ with the right restriction to $G^{\mathbf{d}}_{\mathrm{red}}$, and taking $\lambda_0 = 0$ obviously minimizes $\|\lambda_0 + 2\rho_{\mathbf{d}}\|^2$ (the above inequality becomes an equality). We therefore have $\omega_0 = 2\rho_{\mathbf{d}}$. What does $2\rho_{\mathbf{d}}$ look like? For each $GL(j_i)$ factor of $L_{\mathbf{d}}$, we get a part of $2\rho_{\mathbf{d}}$ that looks like

$$(j_i - 1, j_i - 3, \ldots, 1 - j_i).$$

Thus, in the total $2\rho_{\mathbf{d}}$, a particular coordinate a with $a \geq 0$ occurs once for each factor $GL(j_i)$ with $j_i = a + 2k + 1$ for some k with $k \geq 0$. It follows that the length of block B_a is precisely $\sum_{k\geq 0} \mu_{a+2k+1}$, as desired. It is also clear that the length of block B_a and block B_{-a} are equal.

Next, we consider the case $p \neq 0$. We will consider the first factor $GL(a_1)^{k_1}_{\Delta}$ of $G^{\mathbf{d}}_{\mathrm{red}}$ individually; the other factors would be treated identically. The factor $GL(a_1)^{k_1}_{\Delta}$ of $G^{\mathbf{d}}_{\mathrm{red}}$ sits diagonally across various $GL(j_i)^{b_i}$ factors of $L_{\mathbf{d}}$; indeed, it sits across k_1 of of them. Now, given a weight λ of $L_{\mathbf{d}}$, we obtain the coordinates of the restriction $\lambda|_{G^{\mathbf{d}}_{\mathrm{red}}}$ by summing up coordinates of λ according to the diagonal embedding of the factors of $G^{\mathbf{d}}_{\mathrm{red}}$ in $L_{\mathbf{d}}$. (See [1] for a detailed account of how to restrict weights from $L_{\mathbf{d}}$ to $G^{\mathbf{d}}_{\mathrm{red}}$.) In any case, we add up k_1 distinct coordinates of λ to obtain each coordinate of the $GL(a_1)_\Delta^{k_1}$ part of the restriction of λ. Now, the highest (and only) weight of $G_{\mathrm{red}}^{\mathbf{d}}$ on the representation $\det^{p/c}$ is

$$\Bigg(\underbrace{\Big(\frac{pk_1}{c},\dots,\frac{pk_1}{c}\Big)}_{GL(a_1)_\Delta^{k_1}},\dots,\underbrace{\Big(\frac{pk_l}{c},\dots,\frac{pk_l}{c}\Big)}_{GL(a_l)_\Delta^{k_l}}\Bigg);$$

in particular, every coordinate in the $GL(a_1)_\Delta^{k_1}$ part of the weight is equal to pk_1/c. If we want $\|\lambda\|^2$ to be minimal, it is clear that the k_1 coordinates we add up to obtain this coordinate should consist of pk_1/c 1's and $(c-p)k_1/c$ 0's.

Repeating this argument for every factor of $G_{\mathrm{red}}^{\mathbf{d}}$, we see that $\|\lambda\|^2$ is minimized if, among all its coordinates, there are pn/c 1's and $(c-p)n/c$ 0's. But to compute γ, we need to minimize $\|\lambda+2\rho_{\mathbf{d}}\|^2$, not $\|\lambda\|^2$. We have

$$\|\lambda+2\rho_{\mathbf{d}}\|^2=\|\lambda\|^2+\|2\rho_{\mathbf{d}}\|^2+2\langle\lambda,2\rho_{\mathbf{d}}\rangle.$$

So among possible λ's of minimal size, we could try to choose one so as to minimize $\langle\lambda,2\rho_{\mathbf{d}}\rangle$. In fact, we can arrange for the latter inner product to be 0.

Among the factors $GL(j_i)^{b_i}$ of $L_{\mathbf{d}}$, take the weight $(1,\dots,1)$ (*i.e.*, the determinant character) on pb_i/c of the factors, and $(0,\dots,0)$ on the remaining $(c-p)b_i/c$ of them. Concatenating these weights together, we obtain a weight λ_p of $L_{\mathbf{d}}$. It is easy to see that λ_p has the right number of 0's and 1's to have the desired restriction to $G_{\mathrm{red}}^{\mathbf{d}}$ as well as to be of minimal size. Moreover, as promised, we have that $\langle\lambda_p,2\rho_{\mathbf{d}}\rangle=0$.

Therefore, $\omega_p=\lambda_p+2\rho_{\mathbf{d}}$. This looks very similar to ω_0, except that in each block B_a, a proportion p/c of the coordinates have been increased by 1. Thus ω_p has precisely the desired form. □

3. A conjecture about Dynkin diagrams

Let ${}^LG$, ${}^LB$, ${}^LT$ be the data of the Langlands dual group corresponding to G, B, T, respectively. Let ${}^L\mathfrak{g}$, ${}^L\mathfrak{b}$, ${}^L\mathfrak{h}$ denote the Lie algebras of ${}^LG$, ${}^LB$, ${}^LT$, respectively. The weights of T can be identified with the elements $h\in{}^L\mathfrak{h}$ such that $\alpha^\vee(h)$ is integral for all coroots $\alpha^\vee$ of G (which are the roots of ${}^LG$). This identification allows us to associate to a nilpotent orbit ${}^L\mathcal{O}$ in ${}^L\mathfrak{g}$ a dominant weight for G. Namely, we can choose $e\in{}^L\mathcal{O}$ and let e,h,f span an $\mathfrak{sl}_2$-subalgebra of ${}^L\mathfrak{g}$ with $h\in{}^L\mathfrak{h}$. Then by Dynkin-Kostant theory, h is well-defined up to W-conjugacy, and by $\mathfrak{sl}_2$-theory, h takes integral values at the coroots of G. Hence, h determines an element of Λ_+. We refer to the dominant weight of G thus obtained as the Dynkin weight of ${}^L\mathcal{O}$, and we denote by $\mathcal{D}\subset\Lambda_+$ the set of Dynkin weights of all nilpotent orbits in ${}^L\mathfrak{g}$.

In the previous section, we saw that $\omega_0=2\rho_{\mathbf{d}}$ (made dominant for $GL(n)$). This is nothing more than the Dynkin weight of the nilpotent orbit $\mathcal{O}_{\mathbf{d}^t}$ (we are identifying G with ${}^LG$), since $\mathcal{O}_{\mathbf{d}^t}$ intersects the Lie algebra of $L_{\mathbf{d}}$ in the regular orbit. This result and a similar one for Richardson orbits in other groups (along with calculations in groups of low rank) have led a number of people to conjecture that $\mathcal{D}$ is a subset of Λ_+^{loc} (see [4]). We wish to state a precise conjecture about how $\mathcal{D}$ sits inside of Λ_+^{loc}. To this end, we assign to each ${}^L\mathcal{O}$ in ${}^L\mathfrak{g}$ a finite cover of $\mathcal{O}=d({}^L\mathcal{O})$, where $\mathcal{O}$ is the nilpotent orbit of $\mathfrak{g}$ dual to ${}^L\mathcal{O}$ under Lusztig-Spaltenstein duality. Our conjecture essentially says that if we write our putative finite cover of $\mathcal{O}$ as G/K, then for some τ which is trivial on K, $\gamma(\mathcal{O},\tau)$ equals the Dynkin weight of ${}^L\mathcal{O}$.

Let $A(\mathcal{O})$ denote the fundamental group of $\mathcal{O}$ and let $\bar{A}(\mathcal{O})$ denote Lusztig's canonical quotient of $A(\mathcal{O})$ (see [8], [14]). Let $\mathcal{N}_{o,c}$ be the set of pairs $(\mathcal{O},C)$ consisting of a nilpotent orbit $\mathcal{O}\in\mathfrak{g}$ and a conjugacy class $C\subset A(\mathcal{O})$. We denote by ${}^L\mathcal{N}_o$ the set of nilpotent orbits in ${}^L\mathfrak{g}$. In [14] a duality map $d:\mathcal{N}_{o,c}\to {}^L\mathcal{N}_o$ is defined which extends Lusztig-Spaltenstein duality. This map is surjective and the image of an element $(\mathcal{O},C)\in\mathcal{N}_{o,c}$, denoted $d_{(\mathcal{O},C)}$, depends only on the image of the conjugacy class C in $\bar{A}(\mathcal{O})$.

Now given ${}^L\mathcal{O}\in {}^L\mathfrak{g}$, Proposition 13 of [14] exhibits an explicit element $(\mathcal{O},C)\in\mathcal{N}_{o,c}$ such that $d_{(\mathcal{O},C)}={}^L\mathcal{O}$. The orbit $\mathcal{O}$ also satisfies $\mathcal{O}=d({}^L\mathcal{O})$ (in particular, $\mathcal{O}$ is special). Consider the image of C in $\bar{A}(\mathcal{O})$, which we will also denote by C. We suspect that this conjugacy class coincides with one that Lusztig associates to ${}^L\mathcal{O}$ using the special piece for ${}^L\mathcal{O}$ (see Remark 14 in [14]).

The canonical quotient $\bar{A}(\mathcal{O})$ of $A(\mathcal{O})$ is always of the form S_3,S_4,S_5 or a product of copies of S_2. Hence, it is possible to describe $\bar{A}(\mathcal{O})$ as a Coxeter group of type A and then to associate to each conjugacy class C in $\bar{A}(\mathcal{O})$ a subgroup H_C of $\bar{A}(\mathcal{O})$ which is well-defined up to conjugacy in $\bar{A}(\mathcal{O})$. Lusztig did this for the exceptional groups in [10] and we now do it for the classical groups.

First we need to describe $\bar{A}(\mathcal{O})$ as a Coxeter group in the classical groups (where $\bar{A}(\mathcal{O})$ is a product of copies of S_2). We use the description of $\bar{A}(\mathcal{O})$ in [14]. Let $\lambda=[\lambda_1^{a_1},\lambda_2^{a_2},\dots,\lambda_k^{a_k}]$ be the partition corresponding to $\mathcal{O}$ in the appropriate classical group of type B,C, or D. Let $\mathcal{M}$ be the set of integers m equal to some λ_i such that

$$\begin{gathered}\lambda_i \text{ is odd and } \nu_i \text{ is odd in type } B_n\\ \lambda_i \text{ is even and } \nu_i \text{ is even in type } C_n\\ \lambda_i \text{ is odd and } \nu_i \text{ is even in type } D_n\end{gathered}\tag{2}$$

where $\nu_i=\sum_{j=1}^i a_j$. Then from section 5 of [14], we know that the elements of $\bar{A}(\mathcal{O})$ are indexed by subsets of $\mathcal{M}$ in type C and by subsets of $\mathcal{M}$ of even cardinality in types B and D. In type C we choose our set of simple reflections in $\bar{A}(\mathcal{O})$ to correspond to subsets of $\mathcal{M}$ with a single element. In type B and D we choose our set of simple reflections in $\bar{A}(\mathcal{O})$ to correspond to subsets $\{a,b\}$ of $\mathcal{M}$ with $a>b$ and where no element of $\mathcal{M}$ is both less than a and greater than b. Thus given a conjugacy class C of $\bar{A}(\mathcal{O})$ (which consists of a single element, w, since the group is abelian), we can write w minimally as a product of simple reflections. The simple reflections used are unique, and we define H_C to be the subgroup of $\bar{A}(\mathcal{O})$ generated by those simple reflections. Consider the surjection $\pi:G^e\to\bar{A}(\mathcal{O})$ where $e\in\mathcal{O}$ and define $K=\pi^{-1}(H_C)$ in G^e. We can now make our conjecture.

Given ${}^L\mathcal{O}$ in ${}^L\mathfrak{g}$, we have assigned a conjugacy class C in $\bar{A}(\mathcal{O})$ where $\mathcal{O}=d({}^L\mathcal{O})$ and then a subgroup K in G^e where $e\in\mathcal{O}$. Consider the finite cover $\tilde{\mathcal{O}}=G/K$ of $\mathcal{O}$. Let $\mathbb{C}[\tilde{\mathcal{O}}]$ denote the global algebraic functions on $\tilde{\mathcal{O}}$. It is immediate that $\mathbb{C}[\tilde{\mathcal{O}}]=\sum\Gamma(\mathcal{O},\mathcal{L})$ where the sum is over the irreducible local systems $\mathcal{L}$ (counted with multiplicity) which arise from the irreducible representations of $A(\mathcal{O})$ appearing in $\operatorname{Ind}_{(G^e)^0}^{K}\mathbb{C}$, where $(G^e)^0$ is the identity component of G^e. Hence (1) implies that as a G-module $\mathbb{C}[\tilde{\mathcal{O}}]$ can be written as $\sum_{\lambda\in\Lambda_+}m_\lambda\operatorname{Ind}_T^G\lambda$. Let μ be a weight of largest length with $m_\mu\neq 0$.

Conjecture 3.1. *The weight μ is unique and is the Dynkin weight of ${}^L\mathcal{O}$.*

We have verified the conjecture in a number of cases, although a general proof is elusive at the moment.

It seems likely that when τ gives rise to a local system, denoted $\mathcal{L}_\tau$, that $M(e,\tau)$ is just the direct image of $\mathcal{L}_\tau$ from $\mathcal{O}_e$ to the whole nilpotent cone and so $\Gamma(\mathcal{N}, M(e,\tau)) = \Gamma(\mathcal{O}_e, \mathcal{L}_\tau)$ (this would be consistent with Bezrukavnikov's and Ostrik's work). Since $\mathbb{C}[\tilde{\mathcal{O}}] = \sum \Gamma(\mathcal{O}, \mathcal{L})$, Conjecture 3.1 would then state that the Dynkin weight of ${}^L\mathcal{O}$ occurs as $\gamma(\mathcal{O},\tau)$ for some irreducible representation τ of G^e which is trivial on K.

Remark 3.2. Specifying exactly what τ is seems to be more difficult. For example, let ${}^L\mathcal{O}$ be the subregular orbit in type B_n. Then $\mathcal{O} = d({}^L\mathcal{O})$ is the smallest non-zero special orbit in type C_n. This orbit is Richardson, coming from the parabolic subgroup with Levi factor of type C_{n-1}, and the parabolic subgroup gives rise to a 2-fold cover of $\mathcal{O}$. Thus it is clear that the Dynkin weight of ${}^L\mathcal{O}$ comes from this 2-fold cover of $\mathcal{O}$ (which is, in fact, the one specified by our conjecture) since ${}^L\mathcal{O}$ is regular in a Levi factor of type B_{n-1}. However, when n is odd, the Dynkin weight will correspond to the trivial local system on $\mathcal{O}$, but when n is even, the Dynkin weight will correspond to the non-trivial local system on $\mathcal{O}$ (see the calculations in [4]).

4. Cohomology of the associated sub-bundles

For an element $h \in \mathfrak{h}$, we can define a subspace $\mathfrak{n}_h$ of the nilradical $\mathfrak{n}$ of $\mathfrak{b}$ as follows. We set

$$\mathfrak{n}_h = \bigoplus_{\substack{\alpha \text{ a positive root} \\ \alpha(h)\geq 2}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the α-eigenspace of the root α. As in the previous section, we may identify $\Lambda_+({}^LG)$ with a subset of $\mathfrak{h}$. Then for $\lambda \in \Lambda_+({}^LG)$ we get a subspace $\mathfrak{n}_\lambda$ of $\mathfrak{n}$. Our definition is motivated by the fact that if $\lambda \in \mathfrak{h}$ happens to be a Dynkin weight for a nilpotent orbit $\mathcal{O} \in \mathfrak{g}$, then by work of McGovern, $\mathbb{C}[G \times_B \mathfrak{n}_\lambda] \simeq \mathbb{C}[\mathcal{O}]$ [11] and moreover, by work of Hinich and Panyushev, $H^i(G/B, S^j(\mathfrak{n}_\lambda^*)) = 0$ for all $j \geq 0$ and $i > 0$ [7], [13]. Hence, it seems reasonable, especially given Conjecture 3.1, to pick a general element $\lambda \in \Lambda_+^{loc}({}^LG)$ and study $H^i(G/B, S^j(\mathfrak{n}_\lambda^*))$. Note that $\sum_{j\geq 0} H^0(G/B, S^j(\mathfrak{n}_\lambda^*)) \simeq \mathbb{C}[G \times_B \mathfrak{n}_\lambda]$.

We begin with the definition

Definition 4.1. Two B-representations $V, \tilde{V}$ are called G/B-equivalent if

$$H^i(G/B, S^j(V^*)) \simeq H^i(G/B, S^j(\tilde{V}^*)).$$

for all $i, j \geq 0$.

Our main result is for $GL(n)$, and we identify G with ${}^LG$. We hope in future work to say something interesting for other groups.

Theorem 4.2. *Fix the partition $\mathbf{d}$ and let $\mathfrak{n}_{\mathbf{d},p} = \mathfrak{n}_{\omega_p}$. For any p the B-representations $\mathfrak{n}_{\mathbf{d},p}$ are all mutually G/B-equivalent to each other. Thus, we have for any p that*

$$H^i(G/B, S^j(\mathfrak{n}_{\mathbf{d},p}^*)) \simeq H^i(G/B, S^j(\mathfrak{n}_{\mathbf{d},0}^*))$$

and the latter equals 0 *if* $i > 0$ *and equals* $\mathbb{C}^j[\mathcal{O}_{\mathbf{d}^{\mathrm{t}}}]$ *if* $i = 0$ *(the algebraic functions on the dual orbit of degree* $2j$*).*

The last sentence is just a re-formulation of the Hinich, McGovern, and Panyushev results. Before proving the theorem, we prove two lemmas which rely on the following proposition.

Proposition 4.3. *Let* $\tilde{V} \subset V$ *be representations of* B *such that* $V/\tilde{V} \simeq \mathbb{C}_\mu$ *is a one-dimensional representation of* B *corresponding to the character* μ*. Let* α *be a simple root, and let* P_α *be the parabolic subgroup containing* B *and having* α *as the only positive root in its reductive part. If* V *extends to a* P_α*-representation and* $\langle \alpha^\vee, \mu\rangle = -1$*, then* $V, \tilde{V}$ *are* G/B*-equivalent.*

Proof. Consider the exact sequence $0 \to \tilde{V} \to V \to \mathbb{C}_\mu \to 0$ and its dual $0 \to \mathbb{C}_{-\mu} \to V^* \to \tilde{V}^* \to 0$. By Koszul, we have $0 \to S^{j-1}(V^*) \otimes \mathbb{C}_{-\mu} \to S^j(V^*) \to S^j(\tilde{V}^*) \to 0$ is exact for all $j \geq 0$. By the lemma of Demazure [6], $H^*(G/B, S^{j-1}(V^*) \otimes \mathbb{C}_{-\mu}) = 0$ for all $j \geq 1$ since V^* is P_α-stable and $\langle \alpha^\vee, -\mu\rangle = 1$ (here our B corresponds to the positive roots, hence the difference with Demazure's convention). The result follows from the long exact sequence in cohomology. □

We need to introduce notation to describe the B-stable subspaces of $\mathfrak{n}$. It is clear that if $\mathfrak{g}_\alpha$ belongs to a B-stable subspace U of $\mathfrak{n}$, then so does $\mathfrak{g}_\beta$ for all positive roots β with $\alpha \preceq \beta$ (where $\preceq$ denotes the usual partial order on positive roots). Hence it is enough to describe U by the positive roots α such that $\mathfrak{g}_\alpha \subset U$ and α is minimal among all positive roots with this property. In this case, we say that α is minimal for U.

List the simple roots of G as $\alpha_1, \ldots, \alpha_{n-1}$. Then any positive root of $G = GL(n)$ is of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, which we denote by $[i, j]$. We can express the usual partial order on the positive roots as $[i, j] \preceq [i', j']$ if and only if $i' \leq i$ and $j \leq j'$. We can then specify U by its minimal positive roots, namely a collection of intervals $[i, j]$ such that for any two intervals $[i, j]$ and $[i', j']$ with $i \leq i'$, we have $j \geq j'$. We will say that U is partially specified by the interval $[i, j]$ if the root $[i, j]$ is minimal for U (although there may be other minimal roots). Let us also say that U is i-stable if U is stable under the action of the parabolic subgroup P_{α_i}.

Let U be a B-stable subspace of $\mathfrak{n}$ which is partially specified by the interval $[a, b]$ and which is either $(a-1)$-stable or $(b+1)$-stable. Let U' be the subspace of $\mathfrak{n}$ which is specified by the same intervals as U except that $[a, b]$ is replaced by the two intervals $[a-1, b]$ and $[a, b+1]$. Then U and U' are G/B-equivalent. This is simply an application of Proposition 4.3 where μ is the root $[a, b]$ and α is either α_{a-1} or α_{b+1}. We refer to the G/B-equivalence of U and U' as the *basic move* for $a-1$ (respectively, for $b+1$). We now state two lemmas which rely solely on the basic move.

Lemma 4.4. *Let* U *be a subspace of* $\mathfrak{n}$ *which is partially specified by an interval* $[a, b]$ *and such that* U *is* i*-stable for* $b < i < d$ *for some* d*. Let* U' *be the subspace of* $\mathfrak{n}$ *specified by the same intervals as those defining* U*, but replacing* $[a, b]$ *with the collection of intervals*

$$\{[a - j + 1, d - j] \mid 1 \leq j \leq d - b\}.$$

Then U *is* G/B*-equivalent to* U'*.*

Proof. Applying the basic move for $b+1$ replaces $[a,b]$ with the intervals $[a-1,b]\cup[a,b+1]$. Now apply the basic move repeatedly on the right (for i in the range $b+2\le i<d$) to the interval $[a,b+1]$ and we are left with the two intervals $[a-1,b]\cup[a,d-1]$. The general result follows by induction on $d-b$: we apply the proposition to the interval $[a-1,b]$ with d replaced by $d-1$. The base case $d-b=1$ is trivially true. □

Lemma 4.5. *Let U be a subspace of $\mathfrak{n}$ partially specified by the intervals*

$$[b_0,b_1]\cup[b_1+1,b_2]\cup\cdots\cup[b_{l-1}+1,b_l]\cup[b_l+1,b_{l+1}]$$

and

$$[b_1,b_2-1]\cup\cdots\cup[b_{l-1},b_l-1]$$

where $b_j\le b_{j+1}-2$ for $1\le j\le l-1$. Assume that no interval of U is of the form $[b_l,a_1]$ but that there are intervals of U of the form $[b_0-1,a_2]$ and $[a_3,b_{l+1}+1]$. Let U' be the subspace of $\mathfrak{n}$ specified by the same intervals as U, except that $[b_0,b_1]$ is replaced by $[b_0,b_1-1]$ and $[b_l+1,b_{l+1}]$ is replaced by $[b_l,b_{l+1}]$. Then U is G/B-equivalent to U'. A similar statement holds even if the we omit the interval $[b_l+1,b_{l+1}]$ or $[b_0,b_1]$ from the specification of U.

Proof. Let U_1 be specified by the same intervals as U, except with the above intervals replaced by the intervals

$$[b_0,b_1]\cup[b_1+1,b_2-1]\cup[b_2+1,b_3-1]\cup\cdots\cup[b_{l-1}+1,b_l-1]\cup[b_l+1,b_{l+1}]$$

Then U_1 is seen to be G/B-equivalent to U by applying the the basic move to the roots $[b_1+1,b_2-1],[b_2+1,b_3-1],\dots,[b_{l-1}+1,b_l-1]$ since U_1 is stable for b_2, b_3, … , b_l. The stability for b_l follows from the assumption that no interval of U is of the form $[b_l,a_1]$.

Let U_2 be specified by the same intervals as U_1, except we replace the interval $[b_0,b_1]$ with $[b_0,b_1-1]$ and the interval $[b_l+1,b_{l+1}]$ with $[b_l,b_{l+1}]$. Then U_2 is G/B-equivalent to U_1. This can be seen by applying the basic move to $[b_0,b_1-1]$ (as U_2 is stable for b_1) and applying the basic move (in reverse) to $[b_l,b_{l+1}]$ (as U_1 is b_l-stable). Here we are using the fact that U contains intervals of the form $[b_0-1,a_2]$ and $[a_3,b_{l+1}+1]$.

The proof is completed by observing that U_2 is stable for b_1, b_2, …, b_{l-1}, so we can apply the basic move to the roots $[b_1+1,b_2-1],[b_2+1,b_3-1]$, …, $[b_{l-1}+1,b_l-1]$, arriving at U'. □

Proof of Theorem 4.2. For an orbit corresponding to $\mathbf{d}$ and a local system corresponding to p, we have computed ω_p in Section 2. We recall that c is the g.c.d. of the parts of $\mathbf{d}$ and m_a was defined by the equation $m_ac=\sum_{i\ge0}\mu_{a+2i+1}$, where μ_i is the multiplicity of i as a part in $\mathbf{d}^{\mathrm{t}}$. Write k for one less than the largest part of $\mathbf{d}^{\mathrm{t}}$, and set $s_i=\sum_{j\ge i}m_jc$.

Then by Proposition 2.2, $\mathfrak{n}_{\mathbf{d},p}=\mathfrak{n}_{\omega_p}$ is specified by the set of intervals

$$I_i=[s_{i+1}+pm_i,s_i+pm_{i-1}]$$

where $k\ge i\ge -k+1$. The difference between the left endpoint of I_i and the right endpoint of I_{i+2} will be denoted Δ_i. So

$$\Delta_i=(c-p)m_{i+1}+pm_i.$$

For $i\ge1$ we have $\Delta_i\le\Delta_{-i}\le\Delta_{i-2}$ since $m_{-i}=m_i$ and $m_i\le m_{i-2}$ when $i\ge1$.

Step 1. Application of Lemma 4.4 to each of the intervals I_i shows that $\mathfrak{n}_{\mathbf{d},p}$ is G/B-equivalent to the subspace W of $\mathfrak{n}$ defined by the set of intervals

$$I_{i,j} = [s_{i+1} + pm_i - j + 1, s_{i-1} + pm_{i-2} - j]$$

where $1 \leq j \leq \Delta_i$ if $k \geq i \geq 1$ and where $1 \leq j \leq \Delta_{i-2}$ if $0 \geq i \geq -k+1$. In the former case, if $\Delta_i < j \leq \Delta_{i-2}$, then the intervals $I_{i,j}$ are not minimal, so we may omit them from our specification of W.

Step 2. Now for each r in the range $k \geq r \geq 1$, starting with $r = k$ and working down to $r = 1$, we will modify the intervals $I_{r+1,j}$ and $I_{-r+1,j}$ for $j > pm_{r+1}$ (in the cases where those intervals are defined) to obtain a new subspace of $\mathfrak{n}$ which is G/B-equivalent to W.

First, we will modify $I_{-r+1,j}$ for $\Delta_{r+1} < j \leq \Delta_{-r-1}$. Consider the intervals

$$I_{r-1,j} \cup I_{r-3,j} \cup \cdots \cup I_{-r+5,j} \cup I_{-r+3,j}$$

for $\Delta_{-r-1} < j \leq \Delta_{r-1}$ and the intervals

$$I_{r-1,j} \cup I_{r-3,j} \cup \cdots \cup I_{-r+3,j} \cup I_{-r+1,j}$$

for $\Delta_{r+1} < j \leq \Delta_{-r-1}$. These yield a situation where we can apply Lemma 4.5 a total number of $\Delta_{r-1} - \Delta_{-r-1} = p(m_{r-1} - m_{-r-1})$ times to each of the intervals $I_{-r+1,j}$ for $\Delta_{r+1} < j \leq \Delta_{-r-1}$. This will replace $I_{-r+1,j} = [s_{-r+2} + pm_{-r+1} - j + 1, s_{-r} + pm_{-r-1} - j]$ with $[s_{-r+2} + pm_{-r-1} - j + 1, s_{-r} + pm_{-r-1} - j]$.

Second, we will modify both $I_{r+1,j}$ and $I_{-r+1,j}$ for $pm_{r+1} < j \leq \Delta_{r+1}$ Consider the intervals

$$I_{r-1,j} \cup I_{r-3} \cup \cdots \cup I_{-r+5,j} \cup I_{-r+3,j}$$

for $\Delta_{r+1} < j \leq (\Delta_{r-1} - \Delta_{-r-1}) + \Delta_{r+1}$ and the intervals

$$I_{r+1,j} \cup I_{r-1,j} \cup \cdots \cup I_{-r+3,j} \cup I_{-r+1,j}$$

for $pm_{r+1} < j \leq \Delta_{r+1}$. We again apply Lemma 4.5 a total number of $\Delta_{r-1} - \Delta_{-r-1}$ times by modifying the pair of intervals $I_{r+1,j} \cup I_{-r+1,j}$. Then for $pm_{r+1} < j \leq \Delta_{r+1}$ we replace $I_{r+1,j} \cup I_{-r+1,j}$ with $[s_{r+2} + pm_{r+1} - j + 1, s_r + pm_{r+1} - j] \cup [s_{-r+2} + pm_{-r-1} - j + 1, s_{-r} + pm_{-r-1} - j]$.

We do this for all r starting with $r = k$ and working backward to $r = 1$ (note we haven't done anything to the intervals $I_{1,j}$).

Step 3. At this point our subspace of $\mathfrak{n}$ is specified by the following intervals: $[s_i - l + 1, s_{i-2} - l]$ for $i \geq 2$ with $1 \leq l \leq (c-p)m_i$ and for $1 \geq i$ with $1 \leq l \leq (c-p)m_{i-2}$, and those $I_{i,j}$ where $j \leq pm_i$ for $i \geq 1$ and $j \leq pm_{i+2}$ for $i \leq 0$.

Fix $r \geq 1$. We may assume by induction on r that our subspace is G/B-equivalent to one partially specified by the following intervals (the case $r = 1$ is already true): $[s_i - l + 1, s_{i-2} - l]$ for $1 \leq i \leq r$ and $1 \leq l \leq cm_i$; and $[s_i - l + 1, s_{i-2} - l]$ for $0 \geq i \geq -r+2$ and $1 \leq l \leq cm_{i-2}$. We want to show that the our subspace is G/B-equivalent to one partially specified by the previous intervals together with the cases where $i = r+1$ and $i = -r+1$ in the above formulas.

Let $J_{i,l} = [s_{i+1} - l + 1, s_{i-1} - l]$. Consider the intervals

$$I_{r+1,j} \cup J_{r-2,j+(c-p)m_{r-1}} \cup \cdots \cup J_{-r+2,j+(c-p)m_{r-1}} \cup I_{-r+1,j}$$

for $1 \leq j \leq pm_{r+1}$ and the intervals

$$J_{r-2,l} \cup \cdots \cup J_{-r+2,l}$$

for $1 \leq l \leq (c-p)m_{r-1}$. We apply Lemma 4.5 a total number of $(c-p)(m_{r-1}-m_{r+1})$ times to the pair of intervals $I_{r+1,j} \cup I_{-r+1,j}$ for $1 \leq j \leq pm_{r+1}$. We thus replace $I_{r+1,j} \cup I_{-r+1,j}$ with $[s_{r+2}+pm_{r+1}-j+1, s_{r-1}+(p-c)m_{r+1}-j] \cup [s_{-r+1}+(p-c)m_{-r-1}-j+1, s_{-r}+pm_{-r-1}-j]$. These intervals are just $[s_{r+1}-l+1, s_{r-1}-l] \cup [s_{-r+1}-l+1, s_{-r-1}-l]$ for $(c-p)m_{r+1} < l \leq cm_{r+1}$. Hence, by induction on r we see that our original subspace is G/B-equivalent to the subspace specified by the intervals $J_{i,l}$ where $1 \leq l \leq cm_{i+1}$ for $i \geq 1$ and where $1 \leq l \leq cm_{i-1}$ for $i \leq 0$. This subspace is independent of p which completes the proof. $\square$

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