AN ORDER-REVERSING DUALITY MAP FOR CONJUGACY CLASSES IN LUSZTIG'S CANONICAL QUOTIENT

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Abstract. We define a partial order on the set $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ of pairs $(\mathcal{O}, C)$, where $\mathcal{O}$ is a nilpotent orbit and C is a conjugacy class in $\bar{A}(\mathcal{O})$, Lusztig's canonical quotient of $A(\mathcal{O})$. We then construct an order-reversing duality map $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to {}^L\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ that satisfies many of the properties of the original Spaltenstein duality map. This generalizes work of Sommers [16].

1. Introduction

Let G be a connected complex reductive algebraic group, and $\mathfrak{g}$ its Lie algebra. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{g}$; let $\mathcal{N}_{\mathrm{o}}$ be the set of G-orbits in $\mathcal{N}$, and let $\mathcal{N}_{\mathrm{o}}^{\mathrm{sp}} \subset \mathcal{N}_{\mathrm{o}}$ be the set of special orbits. Given a nilpotent orbit $\mathcal{O} \in \mathcal{N}_{\mathrm{o}}$, let $A(\mathcal{O})$ be the component group of the isotropy group of $\mathcal{O}$ in the adjoint group for $\mathfrak{g}$, and let $\bar{A}(\mathcal{O})$ denote Lusztig's canonical quotient of $A(\mathcal{O})$ (see [10] and [16]). Let $\mathcal{N}_{\mathrm{o,c}}$ denote the set of pairs $(\mathcal{O}, C)$, where $\mathcal{O}$ is a nilpotent orbit, and C is a conjugacy class in $A(\mathcal{O})$. We define $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ the same way, except that we take conjugacy classes in $\bar{A}(\mathcal{O})$ rather than $A(\mathcal{O})$. There is a projection $\mathcal{N}_{\mathrm{o,c}} \to \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ arising from the projection $A(\mathcal{O}) \to \bar{A}(\mathcal{O})$. We also have a projection $pr_1 : \mathcal{N}_{\mathrm{o,c}} \to \mathcal{N}_{\mathrm{o}}$, as well as an inclusion $i : \mathcal{N}_{\mathrm{o}} \to \mathcal{N}_{\mathrm{o,c}}$ which pairs $\mathcal{O}$ with the trivial conjugacy class in $A(\mathcal{O})$. Finally, let ${}^LG$ be the Langlands dual group of G, and define ${}^L\mathfrak{g}$, ${}^L\mathcal{N}$, *etc.* correspondingly.

Spaltenstein [17] defined the well-known duality map $d_{\mathrm{LS}} : \mathcal{N}_{\mathrm{o}} \to \mathcal{N}_{\mathrm{o}}$ for nilpotent orbits by an axiomatic approach. Barbasch and Vogan [4] later defined a closely related map $d_{\mathrm{BV}} : \mathcal{N}_{\mathrm{o}} \to {}^L\mathcal{N}_{\mathrm{o}}$ with an elegant geometric construction involving associated varieties of certain Harish-Chandra modules. Both of these maps are order-reversing and have the set of special orbits as thier image; moreover, via the order-preserving bijection between $\mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}$ and ${}^L\mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}$, these maps can be regarded as being the "same." Finally, Sommers [16] has defined a map $d_{\mathrm{S}} : \mathcal{N}_{\mathrm{o,c}} \to {}^L\mathcal{N}_{\mathrm{o}}$ that extends the Barbasch-Vogan map, in that $d_{\mathrm{S}} \circ i = d_{\mathrm{BV}}$. Every orbit in ${}^L\mathcal{N}_{\mathrm{o}}$ occurs in the image of d_{S}. Moreover, Sommers proves ([16], Proposition 15) that if C and C' are two conjugacy classes in $A(\mathcal{O})$ that descend to the same conjugacy class in $\bar{A}(\mathcal{O})$, then $d_{\mathrm{S}}(\mathcal{O}, C) = d_{\mathrm{S}}(\mathcal{O}, C')$. This means that d_{S} can be regarded as a map $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to {}^L\mathcal{N}_{\mathrm{o}}$.

One task we accomplish in this paper is the introduction of a partial order on the set $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$, as follows. We say that $(\mathcal{O}, C) \le (\mathcal{O}', C')$ if

$$\mathcal{O} \le \mathcal{O}' \qquad \text{and} \qquad d_{\mathrm{S}}(\mathcal{O}, C) \ge d_{\mathrm{S}}(\mathcal{O}', C'). \tag{1}$$

A priori, this partial order might not be well-defined: we might have had $d_{\mathrm{S}}(\mathcal{O}, C) = d_{\mathrm{S}}(\mathcal{O}, C')$ even when $C \ne C'$. In the course of this paper, we rectify this by proving a converse to Proposition 15 of [16].

Theorem 1. *Let $C, C' \subset A(\mathcal{O})$ be two conjugacy classes associated to the same orbit. Then $d_{\mathrm{S}}(\mathcal{O}, C) = d_{\mathrm{S}}(\mathcal{O}, C')$ if and only if C and C' have the same image in $\bar{A}(\mathcal{O})$. As a consequence, the partial order (1) on $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ is well-defined.*

The principal aim of this paper is to construct an order-reversing duality map $\bar{d} : \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to {}^L\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ that extends the existing results and satisfies some of the formal properties of Spaltenstein's original duality map. In particular, such a duality map ought to satisfy the partial-order properties of d_{LS} and d_{BV}:

(1) If $(\mathcal{O}, C) \le (\mathcal{O}', C')$, then $\bar{d}(\mathcal{O}, C) \ge \bar{d}(\mathcal{O}', C')$.
(2) $\bar{d}^2(\mathcal{O}, C) \ge (\mathcal{O}, C)$.

It also ought to be compatible with d_{BV} and d_{S} in the appropriate senses.

(3) $pr_1 \circ \bar{d}(\mathcal{O}, C) = d_{\mathrm{S}}(\mathcal{O}, C)$.

Date: 7 March 2002.

Recall, in particular, that Sommers constructs a "canonical inverse" for d_{S} in [16]: this is a right inverse ${}^L\mathcal{N}_{\mathrm{o}} \to \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ to d_{S}; it is conjecturally closely related to a certain map from $\mathcal{N}_{\mathrm{o}}$ to $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ that Lusztig constructs in [13]. Our map $\bar{d}$ ought to respect this canonical inverse in a suitable way.

(4) $\bar{d}(\mathcal{O}, 1)$ coincides with Sommers' canonical inverse for $\mathcal{O}$.

A map satisfying (1)–(4) is called an *extended duality map*. The main result of the paper is the following.

Theorem 2. *There exists an extended duality map* $\bar{d} : \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to {}^L\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$.

In type A, of course, all the $\bar{A}(\mathcal{O})$-groups are trivial, so this theorem does not say anything new. In the other classical groups, the theorem will be proved by giving an explicit algorithmic construction of a particular map $\bar{d}$ that satisfies the axioms. In the exceptional groups, we define $\bar{d}$ simply by tabulating all its values. The map will turn out to be uniquely determined in types G_2, E_6, and E_7, and various classical groups of small rank, but we will not be able to make a stronger uniqueness statement.

We begin our discussion in Section 2 by deducing certain properties that any extended duality map should have, just from the above axioms. In Section 3, we define the combinatorial objects that will be used to work with $\mathcal{N}_{\mathrm{o}}$ and $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ in the classical groups, and we recall various useful facts about them. Serious work on the classical-groups case begins in Section 4, where we give the definitions of $\bar{d}$ and develop some basic techniques for studying it. Section 5 contains the proofs of the main theorems for the classical groups. Section 6 is devoted to explicit calculations of what an extended duality map can look like, both in classical groups of small rank and in all of the exceptional groups. In the case of the latter, these calculations lead to proofs of the main theorems. Finally, in Section 7, we explore how the present work sits in the context of other work on the nilpotent cone, and where a uniqueness statement for extended duality maps might come from.

I would like to thank R. Bezrukavnikov, V. Ginzburg, R. Kottwitz, V. Ostrik, and D. Vogan for helpful conversations. Furthermore, I would especially like to thank E. Sommers. His paper [16] provides the bulk of the motivation for this one; this paper would not have been possible in the absence of the numerous discussions I have had with him on these topics.

2. Formal properties of duality

For now, let us assume the truth of Theorems 1 and 2, and let $\bar{d} : \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to {}^L\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ be an extended duality map. We can deduce some easy properties of the partial order and of $\bar{d}$ directly from the set-up of Section 1. Let us call a pair $(\mathcal{O}, C)$ *special* if it is in the image of $\bar{d}$, and let us denote the set of special pairs in $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ by $\mathcal{N}^{\mathrm{sp}}_{\mathrm{o},\bar{\mathrm{c}}}$.

Proposition 2.1. *We have $\mathcal{O} \le \mathcal{O}'$ in $\mathcal{N}_{\mathrm{o}}$ if and only if $(\mathcal{O}, 1) \le (\mathcal{O}', 1)$ in $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$. Thus, via the imbedding $\mathcal{O} \mapsto (\mathcal{O}, 1)$, the partially ordered set $\mathcal{N}_{\mathrm{o}}$ can be regarded as a subset of $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ with the inherited partial order.*

Proof. From the definition of the partial order, we know that $(\mathcal{O}, 1) \le (\mathcal{O}', 1)$ implies that $\mathcal{O} \le \mathcal{O}'$. For the converse, we need to prove that if $\mathcal{O} \le \mathcal{O}'$, then $d_{\mathrm{S}}(\mathcal{O}, 1) \ge d_{\mathrm{S}}(\mathcal{O}', 1)$. But we know that $d_{\mathrm{S}}(\mathcal{O}, 1) = d_{\mathrm{BV}}(\mathcal{O})$ and $d_{\mathrm{S}}(\mathcal{O}', 1) = d_{\mathrm{BV}}(\mathcal{O}')$, and we further know that $\mathcal{O} \le \mathcal{O}'$ implies $d_{\mathrm{BV}}(\mathcal{O}) \ge d_{\mathrm{BV}}(\mathcal{O}')$. □

Proposition 2.2. *For a fixed orbit $\mathcal{O}$ and any conjugacy class $C \subset \bar{A}(\mathcal{O})$, we have $(\mathcal{O}, 1) \le (\mathcal{O}, C)$.*

Proof. All we have to check is that $d_{\mathrm{S}}(\mathcal{O}, 1) \ge d_{\mathrm{S}}(\mathcal{O}, C)$. In the exceptional groups, we can verify this simply by scanning Sommers' tables of computed values from [16]. In the classical groups, it is an easy computation from Sommers' formulas for d_{S}, which we recall at the end of Section 3. We defer carrying out the computation until then. □

Proposition 2.3. *Regarding the Sommers duality map d_{S} as being a map $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to {}^L\mathcal{N}_{\mathrm{o}}$, we have that $(\mathcal{O}, C) \le (\mathcal{O}', C')$ implies $d_{\mathrm{S}}(\mathcal{O}, C) \ge d_{\mathrm{S}}(\mathcal{O}', C')$. That is, d_{S} is an order-reversing map.*

Proof. This is an obvious consequence of the definition of the partial order on $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$. □

Proposition 2.4. *We have $\bar{d}^3 = \bar{d}$, so that when we restrict to $\mathcal{N}^{\mathrm{sp}}_{\mathrm{o},\bar{\mathrm{c}}}$, the map $\bar{d}$ is an order-reversing bijection between $\mathcal{N}^{\mathrm{sp}}_{\mathrm{o},\bar{\mathrm{c}}}$ and ${}^L\mathcal{N}^{\mathrm{sp}}_{\mathrm{o},\bar{\mathrm{c}}}$, and $\bar{d}^2$ is the identity map. In general, $\bar{d}^2(\mathcal{O}, C)$ is the smallest special conjugacy class that is greater than or equal to $(\mathcal{O}, C)$.*

Proof. Axiom (2) says that $\bar{d}^2(\mathcal{O},C) \geq (\mathcal{O},C)$. Applying $\bar{d}$ to both sides of this, we obtain $\bar{d}^3(\mathcal{O},C) \leq \bar{d}(\mathcal{O},C)$, by axiom (1). But on the other hand, axiom (2) also tells us that $\bar{d}^2(\bar{d}(\mathcal{O},C)) \geq \bar{d}(\mathcal{O},C)$. We conclude that $\bar{d}^3(\mathcal{O},C) = \bar{d}(\mathcal{O},C)$.

For the second part of the proposition, we know that $\bar{d}^2(\mathcal{O},C)$ is special and greater than or equal to $(\mathcal{O},C)$. Suppose, however, that we had a special pair $(\mathcal{O}',C')$ such that $(\mathcal{O},C) < (\mathcal{O}',C') < \bar{d}^2(\mathcal{O},C)$. We have $\bar{d}(\mathcal{O},C) \geq \bar{d}(\mathcal{O}',C')$, whence $\bar{d}^2(\mathcal{O},C) \leq \bar{d}^2(\mathcal{O}',C')$. But since $(\mathcal{O}',C')$ is special, we have $\bar{d}^2(\mathcal{O}',C') = (\mathcal{O}',C')$, so we can deduce that $\bar{d}^2(\mathcal{O},C) \leq (\mathcal{O}',C')$: a contradiction. □

The following reformulation of the last part of the preceding proposition is sometimes useful for computations.

Corollary 2.5. *Suppose $(\mathcal{O}',C') > (\mathcal{O},C)$, such that for any $(\mathcal{O}'',C'') > (\mathcal{O},C)$, we in fact have $(\mathcal{O}'',C'') \geq (\mathcal{O}',C')$. If $(\mathcal{O},C)$ is not special, then $\bar{d}(\mathcal{O},C) = \bar{d}(\mathcal{O}',C')$.*

Proof. Let $(\mathcal{O}_0,C_0) = \bar{d}^2(\mathcal{O}',C')$: this is the smallest special pair that is greater than or equal to $(\mathcal{O}',C')$. The condition relating $(\mathcal{O}',C')$ and $(\mathcal{O},C)$ implies that $(\mathcal{O}_0,C_0)$ is also the smallest special pair greater than or equal to $(\mathcal{O},C)$. Therefore $\bar{d}^2(\mathcal{O},C) = (\mathcal{O}_0,C_0)$, whence

$$\bar{d}(\mathcal{O},C) = \bar{d}^3(\mathcal{O},C) = \bar{d}(\mathcal{O}_0,C_0) = \bar{d}^3(\mathcal{O}',C') = \bar{d}(\mathcal{O}',C'),$$

as desired. □

Proposition 2.6. *Let $\bar{d}^2(\mathcal{O},C) = (\mathcal{O}',C')$. Then we must either have $(\mathcal{O},C) = (\mathcal{O}',C')$, or $\mathcal{O} \neq \mathcal{O}'$.*

In other words, the smallest special pair sitting above a given nonspecial pair is necessarily attached to a different orbit.

Proof. By Proposition 2.4, we know that $d(\mathcal{O},C) = d(\mathcal{O}',C')$. Then, Theorem 1 tells us that if $\mathcal{O} = \mathcal{O}'$, we must also have $C = C'$. □

Proposition 2.7. *Any pair of the form $(\mathcal{O},1)$ is special.*

Proof. Let $(\mathcal{O}',C') = \bar{d}(\mathcal{O},1)$. Axiom (4) says that $(\mathcal{O}',C')$ is Sommers' canonical inverse for $\mathcal{O}$. Combining this with axiom (3), we see that $d(\mathcal{O}',C') = (\mathcal{O},C)$ for some C. That is, $\bar{d}^2(\mathcal{O},1) = (\mathcal{O},C)$. Now the preceding proposition says that we must have $C = 1$. □

We conclude with an observation about how special orbits (in the Lusztig-Spaltenstein sense) fit into this context.

Proposition 2.8. *An orbit $\mathcal{O}$ is special if and only if $\bar{d}(\mathcal{O},1) = (d_{\mathrm{BV}}(\mathcal{O}),1)$.*

Proof. Let $\mathcal{O}' = d_{\mathrm{BV}}(\mathcal{O})$. From axiom (3), we know that $\bar{d}(\mathcal{O},1) = (\mathcal{O}',C)$ for some C, since $d_{\mathrm{S}}(\mathcal{O},1) = d_{\mathrm{BV}}(\mathcal{O})$. We know from the preceding proposition that $\bar{d}^2(\mathcal{O},1) = \bar{d}(\mathcal{O}',C) = (\mathcal{O},1)$; so in particular, $d_{\mathrm{S}}(\mathcal{O}',C) = \mathcal{O}$. Now, $\mathcal{O}$ is special if and only if $d_{\mathrm{BV}}(\mathcal{O}') = \mathcal{O}$, and the latter holds if and only if $d_{\mathrm{S}}(\mathcal{O}',1) = \mathcal{O}$. In view of Theorem 1, this last equation can hold if and only if $C = 1$, as desired. □

Even if $\mathcal{O}$ is a special orbit, we cannot say anything in general about whether $(\mathcal{O},C)$ is a special pair for nontrivial C. The computed examples in Section 6 include instances of both special and nonspecial pairs of this form.

3. Orbits, partitions, and component groups

We spend this section collecting facts and formulas for working with partitions as a way of understanding nilpotent orbits in the classical groups. It is suggested that the reader skip this section, referring back to it only when necessary to find a particular definition or formula.

3.1. **Partitions.** Let $\mathcal{P}(n)$ be the set of partitions of n. For a partition λ, let $|\lambda|$ denote the sum of the parts of λ. We typically write $\lambda = [\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k]$, and we assume $\lambda_k \neq 0$ unless stated otherwise. Sometimes, however, we shall write partitions as follows, using exponents to indicate multiplicities: $[a_1^{p_1}, \ldots, a_k^{p_k}]$, with $a_1 > \cdots > a_k$. Let $r_\lambda(a)$, or simply $r(a)$, denote the multiplicity of a as a part in λ. We define the *height* of a part in a partition to be the number of parts greater than or equal to the given one: $\mathrm{ht}_\lambda(a) = \mathrm{ht}(a) = \sum_{b \geq a} r_\lambda(b)$. Note that this formula makes sense even if a is not a part of λ; *i.e.*, if $r_\lambda(a) = 0$. We shall employ the notion of height in such circumstances from time to time; we may refer to it as "generalized height" to draw attention to the fact that $r_\lambda(a) = 0$. Finally, we write $\#\lambda$ to denote the total number of parts of λ.

For odd n, we write $\mathcal{P}_B(n)$ for the set of partitions in which even parts occur with even multiplicity. For even n, we write $\mathcal{P}_C(n)$ for the set of partitions in which odd parts occur with even multiplicity, and $\mathcal{P}_D(n)$ for the set of partitions in which even parts occur with even multiplicity. Here, the subscript letters correspond to the type of classical Lie group whose nilpotent orbits are indexed by the given set of partitions, with one caveat: very even partitions (those consisting only of even parts with even multiplicity) in type D correspond to two nilpotent orbits. We ignore this fact throughout the paper, because such orbits have trivial $A(\mathcal{O})$-groups, so the duality map we construct here will not have anything new to say about them. We will sometimes write $\mathcal{P}_1(n)$ for $\mathcal{P}_C(n)$, and $\mathcal{P}_0(n)$ for either $\mathcal{P}_B(n)$ or $\mathcal{P}_D(n)$. This will allow us to make concise statements about $\mathcal{P}_\epsilon(n)$ for $\epsilon \in \{0, 1\}$.

If $\lambda = [\lambda_1 \geq \cdots \geq \lambda_k]$, we write $\sigma_j(\lambda)$ for the i-th *partial sum* $\sum_{i=1}^{j} \lambda_i$. Recall the standard partial order on partitions: for $\lambda, \lambda' \in \mathcal{P}(n)$, we say that $\lambda \leq \lambda'$ if we have $\sigma_j(\lambda) \leq \sigma_j(\lambda')$ for all j. Recall also that the closure order on nilpotent orbits coincides with this order on partitions in the classical groups. For a partition λ, let λ^* denote its transpose partition, and let λ_B, λ_C, λ_D denote its B-, C-, and D-collapses respectively, whenever those are defined. (The X-collapse of λ is the unique largest partition λ' such that $\lambda' \leq \lambda$ and $\lambda' \in \mathcal{P}_X(n)$; see [9].) Suppose $\lambda = [\lambda_1 \geq \cdots \geq \lambda_k] \in \mathcal{P}(n)$, and assume that $\lambda_k \neq 0$. We define the following four operations:

$$\lambda^+ = [\lambda_1 + 1 \geq \lambda_2 \geq \cdots \geq \lambda_k] \qquad \lambda^- = [\lambda_1 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k - 1]$$
$$\lambda_+ = [\lambda_1 \geq \cdots \geq \lambda_k \geq 1] \qquad \lambda_- = \lambda^{*-*}$$

(Note that $\lambda_+ = \lambda^{*+*}$ as well.)

Given two partitions λ and μ, we can form their *union* $\lambda \cup \mu$, a partition of $|\lambda| + |\mu|$, by putting $r_{\lambda\cup\mu}(a) = r_\lambda(a) + r_\mu(a)$ for all a. We can also take their *join*, defined by

$$\lambda \vee \mu = (\lambda^* \cup \mu^*)^*.$$

If one thinks of partitions in terms of Young diagrams, the union corresponds to combining the rows of the two diagrams, while the join corresponds to combining their columns. Finally, if $\lambda = [\lambda_1 \geq \cdots \geq \lambda_k]$, we define

$$\chi_j^+(\lambda) = [\lambda_1 \geq \cdots \geq \lambda_j] \qquad \text{and} \qquad \chi_j^-(\lambda) = [\lambda_{j+1} \geq \cdots \geq \lambda_k].$$

Note that $\lambda = \chi_j^+(\lambda) \cup \chi_j^-(\lambda)$ for any j.

Sometimes we will want to restrict the kinds of partitions that we take unions and joins of, in order to have control over what the union or join looks like. Given two partitions λ and μ, let a be the smallest part of λ, and let b be the smallest part of μ. We say that λ is *superior* to μ if $a \geq b$. We say that λ is *evenly* (resp. *oddly*) *superior* to μ if there is an even (resp. odd) number m such that $a \geq m \geq b$.

3.2. **Computing with collapses.** The following observations about collapses will be relied upon heavily when we set about the work of proving the main theorems in Section 5. If λ has k parts, then any collapse λ_X of it must have either k or $k+1$ parts. Moreover, B-partitions necessarily have an odd number of parts, and D-partitions necessarily have an even number, so we can determine exactly how many parts λ_B or λ_D must have (of course, only one of those collapses is defined for any particular λ). Finally, λ_C (when it is defined) must have the same number of parts as λ, because if it had one more, we would have introduced a new part equal to 1, but we cannot create new odd parts when taking a C-collapse.

We will often encounter situations in which we have a partition written as the union or join of two others, and in which we will want to express a certain collapse of λ in terms of collapses of the smaller partitions. The following proposition collects formulas for twelve kinds of joins, and twelve kinds of unions. This table of formulas is certainly sufficient for the calculations in this paper. The author has not bothered to determine whether any of the twenty-four could have been omitted.

Lemma 3.1. *Suppose $\lambda = \lambda' \vee \lambda''$. Let k be the largest part of λ', and let $p = |\lambda'|$. Suppose in addition that $\mu = \mu' \cup \mu''$, that μ' has k parts, and that $|\mu'| = p$. Assume that λ'^* is superior to λ''^*, and that μ' is superior to μ''. The following table expresses various collapses of λ and μ in terms of collapses of the smaller partitions. For any formula containing ${\lambda''}_B$, we must make the additional assumption that λ'^* is oddly superior to λ''^*; for any containing ${\lambda''}_D$, we assume that λ'^* is evenly superior to λ''^*. Similarly, for any formula containing μ'^- and μ''^+, we must assume that μ' is superior to μ''^+.*

	k *even*		k *odd*	
	p *even*	p *odd*	p *even*	p *odd*
λ_B:	${\lambda'^+}_{B-} \vee {\lambda''}_B$	${\lambda'}_B \vee {\lambda''}_D$	${\lambda'^+}_B \vee {\lambda''^-}_C$	${\lambda'}_B \vee {\lambda''}_C$
λ_C:	${\lambda'}_C \vee {\lambda''}_C$	${\lambda'^+}_C \vee {\lambda''^-}_C$	${\lambda'}_C \vee {\lambda''}_D$	${\lambda'^+}_{C-} \vee {\lambda''}_B$
λ_D:	${\lambda'}_D \vee {\lambda''}_D$	${\lambda'^+}_{D-} \vee {\lambda''}_B$	${\lambda'}_D \vee {\lambda''}_C$	${\lambda'^+}_D \vee {\lambda''^-}_C$
μ_B:	${\mu'}_D \cup {\mu''}_B$	${\mu'^-}_D \cup {\mu''^+}_B$	${\mu'^-}_B \cup {\mu''^+}_D$	${\mu'}_B \cup {\mu''}_D$
μ_C:	${\mu'}_C \cup {\mu''}_C$	${\mu'^-}_C \cup {\mu''^+}_C$	${\mu'}_C \cup {\mu''}_C$	${\mu'^-}_C \cup {\mu''^+}_C$
μ_D:	${\mu'}_D \cup {\mu''}_D$	${\mu'^-}_D \cup {\mu''^+}_D$	${\mu'^-}_B \cup {\mu''^+}_B$	${\mu'}_B \cup {\mu''}_B$

Proof. Once one becomes accustomed to the pattern of producing these formulas, it is fairly easy to compute all of them. We will work through just one: that for λ_B when k is odd and p is even. For λ_B to be defined, $|\lambda|$ must be odd; and since p is even, $|\lambda''|$ must be odd. Since k is odd, the parities of parts of λ'' are opposite to those of the corresponding parts of λ, so taking a B-collapse of λ should manifest itself as something like a C-collapse of λ''. Since $|\lambda''|$ is odd, if we attempt to take a C-collapse of it, we will be partway through a collapsing operation when we get to the end of the partition: there will be a leftover "1" to be added to some odd part, but no remaining odd parts to receive it. This "1" will "leak" onto λ'. We can preemptively take care of this leaking 1 by looking at λ'^+ and λ''^- instead. Now, we comfortably take the C-collapse of λ''^-, and the B-collapse of λ'^+. (If λ'' has m parts, it may seem that we should have added the leaking 1 to the $(m+1)$-th part of λ', not its first part, as is done by writing λ'^+. But in λ'^+, the first part is now even, and the remaining parts up to the m-th one are all odd, so in taking a B-collapse, that "1" gets shoved down to at least the $(m+1)$-th row anyway.) We thus obtain that $\lambda_B = {\lambda'^+}_B \vee {\lambda''^-}_C$.

The only comment we make on other cases is regarding the auxiliary superiority requirements. Terms of the form ${\lambda''}_B$ or ${\lambda''}_D$ may have a different number of parts from λ'', so we have to be a lot more careful in considering the interaction between λ' and λ''. The easiest thing to do is impose a condition that the largest part of λ' have high enough multiplicity that we need not worry: that is exactly what the superiority condition does for us. Similar considerations result in the corresponding requirements when we deal with μ'^- and μ''^+. □

3.3. **Marked partitions.** If X is one of B, C, or D, we define $\tilde{\mathcal{P}}_X(n)$ to be the set of pairs of partitions (ν, η), such that:

(a) $\nu \cup \eta \in \mathcal{P}_X(n)$.
(b) Every part of ν is odd (resp. even) if $X = B$ or D (resp. C) and has multiplicity 1.
(c) If $X = B$ or D, ν has an even number of parts.

This notation is taken from [16], but we will typically find another notation far more convenient for our purposes. We will write elements $(\eta, \nu) \in \tilde{\mathcal{P}}_X(n)$ as ${}^{\langle\nu\rangle}\lambda$, where $\lambda = \nu \cup \eta$. In this notation, we think of elements of $\tilde{\mathcal{P}}_X(n)$ just as partitions from $\mathcal{P}_X(n)$, with the additional data that certain parts (*viz.* those in ν) have been "marked." Indeed, we will refer to elements of these sets as *marked partitions*, and we call ν the *marking partition* and λ the *underlying partition*. Marked partitions of the form ${}^{\langle\varnothing\rangle}\lambda$ are called *trivially marked partitions*. As before, we sometimes write $\tilde{\mathcal{P}}_0(n)$ and $\tilde{\mathcal{P}}_1(n)$ for these sets.

We can attempt to define the union and join operations for marked partitions, but the constructions we give now may not always yield a valid marked partition. This situation will be rectified in the following subsection, when we introduce "reduced marked partitions." For now, we define the union simply by

$$ {}^{\langle\nu_1\rangle}\lambda_1 \cup {}^{\langle\nu_2\rangle}\lambda_2 = {}^{\langle(\nu_1 \cup \nu_2)\rangle}(\lambda_1 \cup \lambda_2). \tag{2}$$

Next, write $\lambda_1 \vee \lambda_2 = [a_1 \geq \cdots \geq a_k]$. Suppose $\nu_1 = [n_1 \geq \cdots \geq n_p]$, and $\nu_2 = [m_1 \geq \cdots \geq m_r]$. We define

$$ {}^{\langle\nu_1\rangle}\lambda_1 \vee {}^{\langle\nu_2\rangle}\lambda_2 = {}^{\langle\omega\rangle}(\lambda_1 \vee \lambda_2), \qquad \text{where} \qquad \omega = [a_{\mathrm{ht}_{\lambda_1}(n_i)} \mid i = 1, \ldots, p] \cup [a_{\mathrm{ht}_{\lambda_2}(m_i)} \mid i = 1, \ldots, r]. \tag{3} $$

The idea of this definition is that we should preserve the heights of the marked parts when we take the join. Quite often, we will encounter joins of marked partitions in which the largest part of λ_2 has very high multiplicity, more than the total number of parts of λ_1. In this special circumstance, understanding the join of marked partitions is much easier: if b is that largest part of λ_2, we obtain

$$\omega = [b + n_1 \geq \cdots \geq b + n_p \geq m_1 \geq \cdots \geq m_r].$$

For $\lambda \in \mathcal{P}_\epsilon(n)$ and $\delta \in \{0, 1\}$, let

$$S_\delta(\lambda) = \{a \mid a \not\equiv \epsilon \pmod 2 \text{ and } r(a) \equiv \delta \pmod 2\}.$$

We will just write S_δ when no confusion will result. For ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_\epsilon(n)$, write

$$T_\delta({}^{\langle\nu\rangle}\lambda) = T_\delta = \nu \cap S_\delta(\lambda).$$

3.4. **Parametrizing $\mathcal{N}_{\mathrm{o,c}}$ and $\mathcal{N}_{\mathrm{o,\bar{c}}}$.** A detailed account of the following description of a parametrization of $\mathcal{N}_{\mathrm{o,c}}$ and $\mathcal{N}_{\mathrm{o,\bar{c}}}$ can be found in [15]. Now, $\tilde{\mathcal{P}}_X(n)$ is close to indexing the set $\mathcal{N}_{\mathrm{o,c}}$ in type X. Actually, there is a surjective map

$$\tilde{\mathcal{P}}_X(n) \to \mathcal{N}_{\mathrm{o,c}}$$

which is a bijection in type B, but is 2-to-1 over any orbit in types C and D whose partition has $S_1 \neq \varnothing$. There is, of course, a further projection

$$\tilde{\mathcal{P}}_X(n) \to \mathcal{N}_{\mathrm{o,\bar{c}}}. \tag{4}$$

We now describe this projection in some detail. Given λ, list the elements of S_1 as $j_l > \cdots > j_1$. Assume that l is even in type C by taking $j_1 = 0$ if necessary (l is automatically odd in type B and even in type D). Now, given ${}^{\langle\nu\rangle}\lambda$, let $T_0^{(m)} = \{a \in T_0({}^{\langle\nu\rangle}\lambda) \mid j_m < a < j_{m+1}\}$, and let $T_1^{(m)} = T_1 \cap \{j_m\}$. Next, we define an equivalence relation $\sim$ on $\tilde{\mathcal{P}}_\epsilon(n)$ as follows: ${}^{\langle\nu\rangle}\lambda \sim {}^{\langle\nu'\rangle}\lambda$ if

(a) $T_0^{(m)}({}^{\langle\nu\rangle}\lambda) = T_0^{(m)}({}^{\langle\nu'\rangle}\lambda)$ whenver m is even.
(b) $\left|T_1^{(m+1)}({}^{\langle\nu\rangle}\lambda) \cup T_0^{(m)}({}^{\langle\nu\rangle}\lambda) \cup T_1^{(m)}({}^{\langle\nu\rangle}\lambda)\right| \equiv \left|T_1^{(m+1)}({}^{\langle\nu'\rangle}\lambda) \cup T_0^{(m)}({}^{\langle\nu'\rangle}\lambda) \cup T_1^{(m)}({}^{\langle\nu'\rangle}\lambda)\right| \pmod 2$ whenever m is odd.

(In the second of these conditions, we interpret $T_1^{(l+1)}$ as $\varnothing$ in type B.) Then, the projection in (4) is precisely the quotient by $\sim$. We can formulate one particular equivalence under $\sim$ quite easily, as follows. If we are working in type B, let $\tilde{S}_1 = S_1 \setminus \{j_l\}$, and note that this set has an even number of elements.

Lemma 3.2. *Given a marked partition ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_X(n)$, define*

$$\nu' = T_0({}^{\langle\nu\rangle}\lambda) \cup \begin{cases} S_1(\lambda) \setminus T_1({}^{\langle\nu\rangle}\lambda) & \text{in types } C \text{ and } D, \\ (\tilde{S}_1(\lambda) \setminus T_1({}^{\langle\nu\rangle}\lambda)) \cup (T_1({}^{\langle\nu\rangle}\lambda) \cap \{j_l\}) & \text{in type } B. \end{cases}$$

Then ${}^{\langle\nu'\rangle}\lambda \sim {}^{\langle\nu\rangle}\lambda$.

Proof. It is easy to see that condition (b) above is satisfied when we replace T_1 by its complement in S_1 in types C and D. In type B, we need to be careful when $m = l$, because there is no j_{l+1}, but the same idea goes through if we take only take the complement of that portion of T_1 which meets $\tilde{S}_1$, as in the above formula. □

Consider the set

$$\tilde{\mathcal{P}}_X^\circ(n) = \left\{{}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_X(n) \;\middle|\; T_1^{(m+1)} = T_0^{(m)} = \varnothing \text{ whenever } m \text{ is odd}\right\},$$

which we call the set of *reduced marked partitions.* It is easy to see that the restricted map $\tilde{\mathcal{P}}_X^\circ(n) \to \mathcal{N}_{\mathrm{o,\bar{c}}}$ is a bijection. An alternate description of these sets is as follows. If λ is of type B (resp. C, D), let us call a part of λ *markable* if it is odd (resp. even, odd) and has odd (resp. even, even) height. Then we have

$$\tilde{\mathcal{P}}_X^\circ(n) = \{{}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_X(n) \mid \nu \text{ consists only of markable parts of } \lambda\}.$$

We will speak of elements of $\tilde{\mathcal{P}}_X(n)$ as *labels* for elements of $\mathcal{N}_{\mathrm{o,c}}$ and $\mathcal{N}_{\mathrm{o,\bar{c}}}$, and of elements of $\tilde{\mathcal{P}}_X^\circ(n)$ as the *reduced labels* for elements of $\mathcal{N}_{\mathrm{o,\bar{c}}}$. Every element of $\tilde{\mathcal{P}}_X(n)$ is $\sim$-equivalent to exactly one element of

$\tilde{\mathcal{P}}^{\circ}_X(n)$. The process of passing to the reduced label can be described as follows. Given ${}^{\langle\nu\rangle}\lambda$, we define a new marked partition ${}^{\langle\nu'\rangle}\lambda$, which is characterized as follows: we have

$$T_0^{(m)}({}^{\langle\nu'\rangle}\lambda) = \begin{cases} T_0^{(m)}({}^{\langle\nu\rangle}\lambda) & \text{if } m \text{ is even} \\ \varnothing & \text{if } m \text{ is odd} \end{cases}$$

and

$$T_1^{(m)}({}^{\langle\nu'\rangle}\lambda) = \begin{cases} \{j_m\} & \text{if } m \text{ is odd and } \left|T_1^{(m+1)}({}^{\langle\nu\rangle}\lambda) \cup T_0^{(m)}(pmp\lambda\nu) \cup T_1^{(m)}({}^{\langle\nu\rangle}\lambda)\right| \text{ is odd} \\ \varnothing & \text{otherwise.} \end{cases}$$

There is often a sensible way, given two arbitrary partitions λ and ν, to carry out a "reduction" procedure that generalizes the above one, even when ${}^{\langle\nu\rangle}\lambda$ is not a valid marked partition. (This goes hand-in-hand with the idea of generalized height.) We must first fix one of the types B, C, or D as the context in which we are working, but we do not require that λ be a partition of that type. The only condition we impose is that when the context type is B or D, ν must have an even number of parts. Note that the above definition of "markable part" makes sense without any restriction on λ, and let $m_k > \cdots > m_1$ be the set of markable parts of λ. We define ν' by putting

$$r_{\nu'}(m_i) = \begin{cases} 1 & \text{if } \mathrm{ht}_\nu(m_i) - \mathrm{ht}_\nu(m_{i+1}) \text{ is odd,} \\ 0 & \text{if } \mathrm{ht}_\nu(m_i) - \mathrm{ht}_\nu(m_{i+1}) \text{ is even,} \end{cases} \qquad r_{\nu'}(a) = 0 \text{ if } a \text{ is not a markable part of } \lambda,$$

where, when $i = k$, we interpret $\mathrm{ht}_\nu(m_{k+1})$ as 0. It is easy to verify that when ${}^{\langle\nu\rangle}\lambda$ is a marked partition, this generalized procedure coincides with the above one for passing to a reduced marked partition.

Let us return to the problem that unions and joins of marked partitions may not yield valid marked partitions. Typically, we employ the above procedure to pass from whatever partitions the formulas (2) and (3) yield to a reduced marked partition. Indeed, henceforth, unless explicitly stated otherwise, all marked partitions are assumed to be reduced, and if any possibly nonreduced marked partition appears in a formula, we silently assume that it is to be replaced by an equivalent reduced one.

3.5. **Duality and special orbits.** We now recall the formulas for d_{LS}, d_{BV}, and d_{S} in the classical groups.

$$\begin{array}{llll} \text{Type } B: & d_{\mathrm{LS}}(\lambda) = \lambda^{*}{}_B & d_{\mathrm{BV}}(\lambda) = \lambda^{-}{}_C{}^{*} & d_{\mathrm{S}}(\nu,\eta) = (\nu \cup \eta^{-}{}_C)^{*}{}_C \\ \text{Type } C: & d_{\mathrm{LS}}(\lambda) = \lambda^{*}{}_C & d_{\mathrm{BV}}(\lambda) = \lambda^{+}{}_B{}^{*} & d_{\mathrm{S}}(\nu,\eta) = (\nu \cup \eta^{+}{}_B)^{*}{}_B \\ \text{Type } D: & d_{\mathrm{LS}}(\lambda) = \lambda^{*}{}_D & d_{\mathrm{BV}}(\lambda) = \lambda^{*}{}_D & d_{\mathrm{S}}(\nu,\eta) = (\nu \cup \eta^{*}{}_D{}^{*})^{*}{}_D \end{array} \tag{5}$$

The formulas for d_{BV} are obtained by combining the formulas for d_{LS} with the following formulas for the order-preserving bijection between $\mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}$ and ${}^L\mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}$ in types B and C:

$$\begin{array}{ll} \mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}(B_n) \to \mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}(C_n): & \lambda \mapsto \lambda^{-}{}_C \\ \mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}(C_n) \to \mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}(B_n): & \lambda \mapsto \lambda^{+}{}_B \end{array}$$

In fact, these same formulas can be evaluated on nonspecial partitions in $\mathcal{N}_{\mathrm{o}}(B_n)$ and $\mathcal{N}_{\mathrm{o}}(C_n)$: they then compute the following composition of maps:

$$\mathcal{N}_{\mathrm{o}} \xrightarrow{d_{\mathrm{LS}}^2} \mathcal{N}_{\mathrm{o}}^{\mathrm{sp}} \xrightarrow{\simeq} {}^L\mathcal{N}_{\mathrm{o}}^{\mathrm{sp}}.$$

The formulas for d_{S} are given in [16]. We are now in a position to revisit the proof of Proposition 2.2.

Proof of Proposition 2.2 in the classical types. We need to show that $d_{\mathrm{S}}(\mathcal{O},1) \le d_{\mathrm{S}}(\mathcal{O},C)$; this should follow from a quick computation using the above formulas. We carry it out now in type D. Starting with $\lambda = \nu \cup \eta$, it is easy to see that

$$\eta \le \eta^{*}{}_D{}^{*} \tag{6a}$$

$$\lambda = \nu \cup \eta \le \nu \cup \eta^{*}{}_D{}^{*} \tag{6b}$$

$$\lambda^{*} \ge (\nu \cup \eta^{*}{}_D{}^{*})^{*} \tag{6c}$$

$$\lambda^{*}{}_D \ge (\nu \cup \eta^{*}{}_D{}^{*})^{*}{}_D. \tag{6d}$$

Essentially the same reasoning works in types B and C as well, although we need to replace (6b) above with the following slightly less trivial inequalities:

$$\lambda^-{}_C \leq \nu \cup \eta^-{}_C, \qquad \lambda^+{}_B \leq \nu \cup \eta^+{}_B.$$

Moreover, in types B and C, we need to use the observations that $\lambda^-{}_C{}^* = \lambda^-{}_C{}^*{}_C$ and $\lambda^+{}_B{}^* = \lambda^+{}_B{}^*{}_B$, respectively, to pass from (6c) to (6d). □

We also recall the recipe for computing Sommers' canonical inverse. if λ is of type B (resp. C or D), we let π be the set of even (resp. odd) parts of λ^* with odd multiplicity. Then the canonical inverse is given by ${}^{\langle\pi\rangle}d_{\mathrm{BV}}(\lambda)$, where we pass to the reduced marked partition if necessary. (In [16], Sommers regards the canonical inverse as a map ${}^L\mathcal{N}_{\mathrm{o}} \to \mathcal{N}_{\mathrm{o,c}}$, so he made no comment about passing a reduced marked partition, but in the present context, we regard it as a map ${}^L\mathcal{N}_{\mathrm{o}} \to \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$.)

The images of d_{LS} and d_{BV} consist precisely of the set of special orbits, which are labelled by special partitions. A characterization of special partitions may be found in [9]. If λ is a B- (resp. C-, D-) partition, it is special if all its even (resp. odd, even) parts have odd (resp. even, even) height. Moreover, if λ is a special B- (resp. C-) partition, then λ^* is also a special B- (resp C-) partition. If λ is a special D-partition, then λ^* is a (not necessarily special) C-partition. We conclude with a lemma about formulas for special partitions.

Lemma 3.3. *The following identities hold:* $\lambda^-{}_C{}^* = \lambda^{*-}{}_C$ *for* $\lambda \in \mathcal{P}_B(n)$*,* $\lambda^+{}_B{}^* = \lambda^{*+}{}_B$ *for* $\lambda \in \mathcal{P}_C(n)$*, and* $\lambda^*{}_D{}^* = \lambda^{+-}{}_C$ *if either* $\lambda \in \mathcal{P}_D(n)$ *or* $\lambda^* \in \mathcal{P}_C(n)$*.*

Proof. The proof establishes all three formulas simultaneously by induction on the sum of the partition. One verifies it by direct calculation for the smallest partitions: $[3]$ and $[1^3]$ in type B, $[2]$ and $[1^2]$ in type C, and $[1^2]$ in type D. We work out the inductive step when λ is of type B; the others are handled similarly. Let $m = \#\lambda$, and let b be the smallest part of λ. Note that m is necessarily odd. We can write $\lambda = [b^m] \vee \lambda'$, where λ' is a B-partition if b is even, and a C-partition if b is odd. Suppose first that b is odd. We have $\lambda^- = [b^m]^- \vee \lambda'$, so $\lambda^-{}_C = [b^m]^-{}_C \vee \lambda'_D$ by Lemma 3.1. Now, $[b^m]^-{}_C = [b^{m-1}, b-1]$, so we get $\lambda^-{}_C{}^* = [m^{b-1}, m-1] \cup \lambda'_D{}^*$. Using the inductive hypothesis, we rewrite this as $[m^b]^- \cup \lambda'^{*+-}{}_C$. Another appeal to Lemma 3.1 lets us conclude that this last expression is equal to $\lambda^{*-}{}_C$. The case of b even is handled similarly, as are types C and D. □

4. Construction in the classical groups

We are now ready to define the map $\bar{d} : \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to {}^L\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$. Since we know that we want $pr_1 \circ \bar{d}$ to agree with d_{S}, the main difficulty is defining the marking partition on the range. Given a reduced marked partition ${}^{\langle\nu\rangle}\lambda$, write $\nu = [n_l > \cdots > n_1]$, and assume that l is even by taking $n_1 = 0$ if necessary in type C. Define

$$\hat{\nu} = [\mathrm{ht}_\lambda(n_1) - 1 > \cdots > \mathrm{ht}_\lambda(n_l) - 1]. \tag{7}$$

(If we are in type C and $n_1 = 0$, we need to say what $\mathrm{ht}_\lambda(0)$ means. We want this quantity to be even, since markable parts are supposed to have even height in type C. We take it to be the smallest even number larger than $\#\lambda$.) Next, if ${}^{\langle\nu\rangle}\lambda = (\nu, \eta)$ is a marked partition of type B (resp. C, D), we define

$$\pi = \{\text{even (resp. odd, odd) parts of } \eta^* \text{ with odd multiplicity}\}. \tag{8}$$

We regard this set as a partition, each of whose parts has multiplicity 1. We then put

$$\begin{aligned} &\textit{Type B:} \quad {}^{\langle\rho\rangle}\tau = {}^{\langle\varnothing\rangle}\nu^* \vee {}^{\langle\pi\rangle}\eta^-{}_C{}^* \qquad \bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\hat{\nu}\cup\rho\rangle}\tau_C \\ &\textit{Type C:} \quad {}^{\langle\rho\rangle}\tau = {}^{\langle\varnothing\rangle}\nu^* \vee {}^{\langle\pi\rangle}\eta^+{}_B{}^* \qquad \bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\hat{\nu}\cup\rho\rangle}\tau_B \\ &\textit{Type D:} \quad {}^{\langle\rho\rangle}\tau = {}^{\langle\varnothing\rangle}\nu^* \vee {}^{\langle\pi\rangle}\eta^*{}_D \qquad \bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\hat{\nu}\cup\rho\rangle}\tau_D. \end{aligned} \tag{9}$$

We must carry out a reduction procedure three times while computing $\bar{d}({}^{\langle\nu\rangle}\lambda)$: once for ${}^{\langle\pi\rangle}\nu^-{}_C{}^*$ and its analogues, once for ${}^{\langle\rho\rangle}\tau$, and once for the final answer. In this and the following section, we will be ensconced in many laborious computations with these formulas. Most of the results must actually be proved thrice, once in each of types B, C, and D; but we will usually only write out the full details in type B, and just make cursory remarks about the nature of the calculations in the other types.

Example 4.1. Consider the orbit $\mathcal{O}$ labelled by $[7,5,4^2,3,2^2,1^2]$ in type B_{14}, or $\mathfrak{so}(29)$. This partition has three markable parts: 7, 3, and 1. Therefore, $\bar{A}(\mathcal{O}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$; the four possible marking partitions are $\varnothing$, $[3,1]$, $[7,3]$, and $[7,1]$. Let us consider the conjugacy class corresponding to $[3,1]$. Writing ${}^{\langle[3,1]\rangle}[7,5,4^2,3,2^2,1^2]$ as a pair, we have $(\nu,\eta) = ([3,1],[7,5,4^2,2^2,1])$. We compute $\eta^-{}_C = [6^2,4^2,2^2]$, which is self-dual: $\eta^-{}_C{}^* = [6^2,4^2,2^2]$. We have $\nu^* = [2,1^2]$, so $\tau = [8,7,5,4,2^2]$. Finally, $\tau_C = [8,6^2,4,2^2]$.

To compute the marking partition, we have $\eta^* = [7,6,4^2,2,1^2]$, so $\pi = [6,2]$. Both parts of π are markable in $\eta^-{}_C{}^*$, so ${}^{\langle\pi\rangle}\eta^-{}_C{}^*$ is already reduced. Taking the join with ν^* yields ${}^{\langle[6,2]\rangle}[8,7,5,4,2^2]$, which becomes ${}^{\langle[4,2]\rangle}[8,7,5,4,2^2]$ when we reduce it. Finally, $\hat{\nu} = [8,4]$, so for the final answer, we take the reduced marked partition corresponding to ${}^{\langle[8,4^2,2]\rangle}[8,6^2,4,2^2]$, arriving at

$$\bar{d}({}^{\langle[3,1]\rangle}[7,5,4^2,3,2^2,1^2]) = {}^{\langle[4,2]\rangle}[8,6^2,4,2^2].$$

We can establish the following two properties of $\bar{d}$ immediately from the definition.

Proposition 4.2. *We have that $pr_1 \circ \bar{d}$ agrees with d_{S}.*

Proof. In type B, the underlying partition of $\bar{d}({}^{\langle\nu\rangle}\lambda)$ is $\tau_C = (\nu^* \vee \eta^-{}_C{}^*)_C = (\nu \cup \eta^-{}_C)^*{}_C$, which is precisely the formula for $d_{\mathrm{S}}(\nu,\eta)$ in type B. Types C and D are equally easy to handle. $\square$

Proposition 4.3. *Given an orbit $\mathcal{O}$ labelled by a partition λ, the conjugacy class labelled by $\bar{d}({}^{\langle\varnothing\rangle}\lambda)$ coincides with Sommers' canonical inverse for $\mathcal{O}$.*

Proof. When $\nu = \varnothing$ and $\eta = \lambda$, the formula for ${}^{\langle\rho\rangle}\tau$ in (9) agrees with Sommers' recipe for the canonical inverse. At this stage, $\tau = d_{\mathrm{BV}}(\lambda)$ is already a C-, B-, or D-partition (in types B, C, and D, respectively), so the additional collapse of τ in the formula for $\bar{d}$ does nothing. We also have $\hat{\nu} = \varnothing$, so $\bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\pi\rangle}\tau$. $\square$

Before we can set about proving that the above map is, in fact, an extended duality map as defined in Section 1, we need to develop some techniques for manipulating marked partitions. The formulas we have so far are too opaque to be tackled in their raw form when we want to prove things about them. We spend the rest of the section showing how to break down a marked partition into "blocks," and how to compute $\bar{d}$ piecemeal on the individual blocks.

Lemma 4.4. *Suppose that ${}^{\langle\nu\rangle}\lambda = {}^{\langle\varnothing\rangle}[a^l] \vee {}^{\langle\nu'\rangle}\lambda'$, where λ has l parts.*

(a) *If ${}^{\langle\nu\rangle}\lambda$ is of type B or D and a is even, then $\bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\varnothing\rangle}[l^a] \cup \bar{d}({}^{\langle\nu'\rangle}\lambda')$. In this case, ${}^{\langle\nu'\rangle}\lambda'$ is of the same type as ${}^{\langle\nu\rangle}\lambda$.*

(b) *If ${}^{\langle\nu\rangle}\lambda$ is of type C, a is odd, and l is even, then $\bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\varnothing\rangle}[l+1,l^{a-1}] \cup \bar{d}({}^{\langle\nu'\rangle}\lambda')$. Here, ${}^{\langle\nu'\rangle}\lambda'$ is of type D.*

In both cases, we also have $\bar{d}^2({}^{\langle\nu\rangle}\lambda) = {}^{\langle\varnothing\rangle}[a^l] \vee \bar{d}^2({}^{\langle\nu'\rangle}\lambda')$.

Proof. Let us write ${}^{\langle\nu\rangle}\lambda = (\nu,\eta)$ and ${}^{\langle\nu'\rangle}\lambda' = (\nu',\eta')$, and let us refer back to the formulas for d_{S}. If η has n parts, then

$$\nu = [a^{l-n}] \vee \nu' \qquad \text{and} \qquad \eta = [a^n] \vee \eta'.$$

We will prove part (a) when ${}^{\langle\nu\rangle}\lambda$ is of type B. The type-D case of part (a), as well as part (b) and the statement for $\bar{d}^2$, are handled similarly. Now, η^- may be given by either $[a^n] \vee \eta'^-$ or $[a^{n-1},a-1] \vee \eta'$, depending on whether η' has n parts or fewer than n parts. We compute $\eta^-{}_C$ with the appropriate formula from Lemma 3.1, and see that in either case, we get $[a^n] \vee \eta'^-{}_C$ (possibly using the fact that $[a^{n-1},a-1]^+{}_C = [a^n]$). Therefore,

$$\nu \cup \eta^-{}_C = ([a^{l-n}] \vee \nu') \cup ([a^n] \vee \eta'^-{}_C) = [a^l] \vee (\nu' \cup \eta'^-{}_C).$$

Taking the transpose of both sides, we get

$$(\nu \cup \eta^-{}_C)^* = [l^a] \cup (\nu' \cup \eta'^-{}_C)^*.$$

Now, we use another formula from Lemma 3.1 to compute the C-collapse of this expression. We obtain

$$d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda) = (\nu \cup \eta^-{}_C)_C = [l^a] \cup (\nu' \cup \eta'^-{}_C)_C = [l^a] \cup d_{\mathrm{S}}({}^{\langle\nu'\rangle}\lambda'). \tag{10}$$

We now need to compute the marking partition. If π is defined from η according to (8), and π' is defined analagously from η', we evidently have $\pi = \pi'$. It is then easy to work through the formulas of (9) and see that $\rho = \rho'$ as well, and finally that $\bar{d}({}^{\langle\nu\rangle}\lambda)$ and $\bar{d}({}^{\langle\nu'\rangle}\lambda')$ have the same marking partition. $\square$

Lemma 4.5. *Given ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_B(n)$, suppose that ${}^{\langle\nu\rangle}\lambda = {}^{\langle\nu_1\rangle}\lambda_1 \cup {}^{\langle\nu_2\rangle}\lambda_2$, with ${}^{\langle\nu_1\rangle}\lambda_1 \in \tilde{\mathcal{P}}_B(m)$ and ${}^{\langle\nu_2\rangle}\lambda_2 \in \tilde{\mathcal{P}}_D(n-m)$. Suppose furthermore that λ_1 is evenly superior to λ_2. Then $\bar{d}({}^{\langle\nu\rangle}\lambda) = \bar{d}({}^{\langle\nu_1\rangle}\lambda_1) \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2)$. (Note that the term $\bar{d}({}^{\langle\nu_2\rangle}\lambda_2)$ is to be computed in type D.)*

Proof. Write ${}^{\langle\nu\rangle}\lambda = (\nu, \eta)$, ${}^{\langle\nu_1\rangle}\lambda_1 = (\nu_1, \eta_1)$, and ${}^{\langle\nu_2\rangle}\lambda_2 = (\nu_2, \eta_2)$ in the notation of pairs, and let π, π_1, and π_2 be the corresponding partitions as defined in (8). Note that η_1 has an odd number of parts, and $|\eta_1|$ is odd, both because η_1 is a B-partition. We consult Lemma 3.1, starting with $\eta^- = \eta_1 \cup \eta_2^-$, and find that $\eta^-{}_C = \eta_1^-{}_C \cup \eta_2^-{}^+{}_C$; then, Lemma 3.3 tells us that we actually have $\eta^-{}_C = \eta_1^-{}_C \cup \eta_2^*{}_D{}^*$. Because η_1 has an odd number of parts, even parts of η^* correspond to odd parts of η_2^*, so π_1 and π_2 are related to π as in the following equation:

$$ {}^{\langle\pi\rangle}\eta^-{}_C{}^* = {}^{\langle\pi_1\rangle}\eta_1^-{}_C{}^* \vee {}^{\langle\pi_2\rangle}\eta_2^*{}_D. $$

Since $\nu = \nu_1 \cup \nu_2$, we have $\nu^* = \nu_1^* \vee \nu_2^*$; it follows directly that

$$ {}^{\langle\varnothing\rangle}\nu^* \vee {}^{\langle\pi\rangle}\eta^-{}_C{}^* = ({}^{\langle\varnothing\rangle}\nu_1^* \vee {}^{\langle\pi_1\rangle}\eta_1^-{}_C{}^*) \vee ({}^{\langle\varnothing\rangle}\nu_2^* \vee {}^{\langle\pi_2\rangle}\eta_2^*{}_D). \tag{11} $$

Write this equation, following (9), as ${}^{\langle\rho\rangle}\tau = {}^{\langle\rho_1\rangle}\tau_1 \vee {}^{\langle\rho_2\rangle}\tau_2$. Now, τ_1 has an odd number of columns, because $\#\lambda_1$ is odd. (One might worry that it could have fewer columns due to the "$^-$" operation, if the smallest part of η_1 were 1, but that is not possible since λ_1 is superior to λ_2.) Now we make use of the hypothesis of even superiority: another appeal to Lemma 3.1 tells us exactly that $\tau_C = \tau_{1C} \vee \tau_{2D}$. At this point, we have established that

$$ d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda) = d_{\mathrm{S}}({}^{\langle\nu_1\rangle}\lambda_1) \vee d_{\mathrm{S}}({}^{\langle\nu_2\rangle}\lambda_2). \tag{12} $$

Now consider the marking partition. If n is a part of ν_2, we have $\mathrm{ht}_\lambda(n) = \mathrm{ht}_{\lambda_2}(n) + \#\lambda_1$. We therefore have $\hat{\nu} = \hat{\nu}_1 \cup ([(\#\lambda_1)^{\#\hat{\nu}_2}] \vee \hat{\nu}_2)$. Combining this description with (12) and (11), we find

$$ {}^{\langle\hat{\nu}\cup\rho\rangle}\tau_C = {}^{\langle\hat{\nu}_1\cup\rho_1\rangle}\tau_{1C} \vee {}^{\langle\hat{\nu}_2\cup\rho_2\rangle}\tau_{2D}, $$

as desired. □

Entirely analagous arguments establish the three cases of the following lemma.

Lemma 4.6. *Let ${}^{\langle\nu\rangle}\lambda$ be a marked partition, and suppose that ${}^{\langle\nu\rangle}\lambda = {}^{\langle\nu_1\rangle}\lambda_1 \cup {}^{\langle\nu_2\rangle}\lambda_2$.*

(a) *If ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_C(n)$, let us also suppose that ${}^{\langle\nu_1\rangle}\lambda_1 \in \tilde{\mathcal{P}}_C(m)$, that ${}^{\langle\nu_2\rangle}\lambda_2 \in \tilde{\mathcal{P}}_C(n-m)$, that λ_1 and ν_1 have an even number of parts, and that λ_1 is oddly superior to λ_2. Then $\bar{d}({}^{\langle\nu\rangle}\lambda) = \bar{d}({}^{\langle\nu_1\rangle}\lambda_1)_- \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2)$. Here, $\bar{d}({}^{\langle\nu_1\rangle}\lambda_1)_-$ is to be understood as applying the $_-$ operation to the underlying partition. The largest part of the underlying partition is odd, so the marking partition is unaffected.*

(b) *If ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_D(n)$, let us also suppose that ${}^{\langle\nu_1\rangle}\lambda_1 \in \tilde{\mathcal{P}}_D(m)$, that ${}^{\langle\nu_2\rangle}\lambda_2 \in \tilde{\mathcal{P}}_D(n-m)$, and that λ_1 is evenly superior to λ_2. Then $\bar{d}({}^{\langle\nu\rangle}\lambda) = \bar{d}({}^{\langle\nu_1\rangle}\lambda_1) \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2)$.*

(c) *In the context of case (a), let us further suppose that λ, ν, λ_2, and ν_2 have even numbers of parts. Then $\bar{d}({}^{\langle\nu\rangle}\lambda)_- = \bar{d}({}^{\langle\nu_1\rangle}\lambda_1)_- \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2)_-$.* □

Case (c) of this lemma may seem bizarre, but we will arrive at a use for it shortly.

Returning to the context of Lemma 4.5, let m be the even number arising in the definition of "evenly superior" for λ_1 and λ_2, and let $l = \#\lambda_1$. (Note that l is odd.) There is marked B-partition ${}^{\langle\nu_1'\rangle}\lambda_1'$ such that ${}^{\langle\nu_1\rangle}\lambda_1 = {}^{\langle\varnothing\rangle}[m^l] \vee {}^{\langle\nu_1'\rangle}\lambda_1'$. Using Lemma 4.4, we can write

$$ \begin{aligned} \bar{d}({}^{\langle\nu\rangle}\lambda) &= ({}^{\langle\varnothing\rangle}[l^m] \cup \bar{d}({}^{\langle\nu_1'\rangle}\lambda_1')) \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2) \\ &= ({}^{\langle\varnothing\rangle}[l^m] \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2)) \cup \bar{d}({}^{\langle\nu_1'\rangle}\lambda_1'), \end{aligned} \tag{13} $$

where we have made use of the fact that the largest part of λ_2 is at most m, so the underlying partition of $\bar{d}({}^{\langle\nu_2\rangle}\lambda_2)$ has at most m parts. We can apply part (b) of Lemma 4.4 to the first term and write

$$ \bar{d}({}^{\langle\varnothing\rangle}[l^m] \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2)) = {}^{\langle\varnothing\rangle}[m+1, m^{l-1}] \cup \bar{d}^2({}^{\langle\nu_2\rangle}\lambda_2). $$

Now, the union in (13) is exactly of the form demanded by part (a) of Lemma 4.6, so we can apply that statement here.

$$\begin{aligned}\bar{d}^2({}^{\langle\nu\rangle}\lambda) &= \bar{d}({}^{\langle\varnothing\rangle}[l^m] \vee \bar{d}({}^{\langle\nu_2\rangle}\lambda_2))_- \vee \bar{d}^2({}^{\langle\nu_1'\rangle}\lambda_1')\\ &= ({}^{\langle\varnothing\rangle}[m^l] \cup \bar{d}^2({}^{\langle\nu_2\rangle}\lambda_2)) \vee \bar{d}^2({}^{\langle\nu_1'\rangle}\lambda_1')\\ &= ({}^{\langle\varnothing\rangle}[m^l] \vee \bar{d}^2({}^{\langle\nu_1'\rangle}\lambda_1')) \cup \bar{d}^2({}^{\langle\nu_2\rangle}\lambda_2).\end{aligned}$$

By one final application of Lemma 4.4, we obtain the following result for type B. Similar calculations establish it in types C and D.

Lemma 4.7. *Let ${}^{\langle\nu\rangle}\lambda = {}^{\langle\nu_1\rangle}\lambda_1 \cup {}^{\langle\nu_2\rangle}\lambda_2$ be a decomposition as in Lemma 4.5 or 4.6. Then $\bar{d}^2({}^{\langle\nu\rangle}\lambda) = \bar{d}^2({}^{\langle\nu_1\rangle}\lambda_1) \cup \bar{d}^2({}^{\langle\nu_2\rangle}\lambda_2)$.* □

Now, we can use Lemma 4.6(b) iteratively to split up a marked D-partition into smaller and smaller pieces. We can also do the same in type B, if we use Lemma 4.5 and Lemma 4.6(b) in combination. Similarly, parts (a) and (c) of Lemma 4.6 taken together let us split up C-partitions into smaller and smaller pieces. The following definition captures the precise nature of the permitted decompositions.

Definition 4.8. Suppose we have ${}^{\langle\nu\rangle}\lambda = {}^{\langle\nu_1\rangle}\lambda_1 \cup \cdots \cup {}^{\langle\nu_k\rangle}\lambda_k$. Such a decomposition is called a *division into blocks* of ${}^{\langle\nu\rangle}\lambda$, and each ${}^{\langle\nu^i\rangle}\lambda_i$ is called a *block*, under the circumstances described below.

If ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_B(n)$, we require that ${}^{\langle\nu_1\rangle}\lambda_1 \in \tilde{\mathcal{P}}_B(k_1)$ and that ${}^{\langle\nu_i\rangle}\lambda_i \in \tilde{\mathcal{P}}_D(k_i)$ for $i > 1$. Furthermore, λ_i must be evenly superior to λ_{i+1} for $i = 1, \ldots, k-1$.

If ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_C(n)$, we require that ${}^{\langle\nu_i\rangle}\lambda_i \in \tilde{\mathcal{P}}_C(k_i)$ for all i, and that λ_i and ν_i have an even number of parts for $i = 1, \ldots, k-1$. Furthermore, λ_i must be oddly superior to λ_{i+1} for $i = 1, \ldots, k-1$.

If ${}^{\langle\nu\rangle}\lambda \in \tilde{\mathcal{P}}_D(n)$, we require that ${}^{\langle\nu_i\rangle}\lambda_i \in \tilde{\mathcal{P}}_D(m)$ for all i. Furthermore, λ_i must be evenly superior to λ_{i+1} for $i = 1, \ldots, k-1$.

We now combine Lemmas 4.5, 4.6, and 4.7 to obtain the following concise statement.

Proposition 4.9. *Let ${}^{\langle\nu\rangle}\lambda = {}^{\langle\nu_1\rangle}\lambda_1 \cup \cdots \cup {}^{\langle\nu_k\rangle}\lambda_k$ be a division into blocks. Then, $\bar{d}$ can be computed as follows:*

Type B: $\quad \bar{d}({}^{\langle\nu\rangle}\lambda) = \bar{d}({}^{\langle\nu_1\rangle}\lambda_1) \vee \cdots \vee \bar{d}({}^{\langle\nu_k\rangle}\lambda_k)$

Type C: $\quad \bar{d}({}^{\langle\nu\rangle}\lambda) = \bar{d}({}^{\langle\nu_1\rangle}\lambda_1)_- \vee \cdots \vee \bar{d}({}^{\langle\nu_{k-1}\rangle}\lambda_{k-1})_- \vee \bar{d}({}^{\langle\nu_k\rangle}\lambda_k)$

Type D: $\quad \bar{d}({}^{\langle\nu\rangle}\lambda) = \bar{d}({}^{\langle\nu_1\rangle}\lambda_1) \vee \cdots \vee \bar{d}({}^{\langle\nu_k\rangle}\lambda_k)$ □

Moreover, in all types, $\bar{d}^2({}^{\langle\nu\rangle}\lambda) = \bar{d}^2({}^{\langle\nu_1\rangle}\lambda_1) \cup \cdots \cup \bar{d}^2({}^{\langle\nu_k\rangle}\lambda_k)$. □

The motivation for developing the idea of divisions into blocks is our hope that we can cut up arbitrary marked partitions into blocks that are very simple in some sense, and that such blocks will be easy to work with when we set about the task of proving the main theorems. We now state precisely the sort of blocks we hope to obtain.

Definition 4.10. A *basic block* of type B (resp. C, D) is a marked partition ${}^{\langle\nu\rangle}\lambda$ such that ν has one or two parts, say $[n_2]$ or $[n_2 > n_1]$, such that n_1 (if it exists) is the smallest part of λ, and such that n_2 is the largest part of odd (resp. even, even) height in λ. The circumstance of ν having only one part can occur only in type C; in this case, we often regard ν as having two parts by putting $n_1 = 0$. A basic block is called *ultrabasic* if it meets the additional condition that $n_1 \leq 1$.

Proposition 4.11. *Any marked partition has a division into blocks such that each block is either a trivially marked partition or a basic block.*

Proof. This is easily seen by induction on the number of parts of the underlying partition. Given a marked partition ${}^{\langle\nu\rangle}\lambda$ of type B, let a be the first part of odd height. If a is even (and therefore unmarkable) or odd and unmarked, we put $\lambda_1 = \chi^+_{\mathrm{ht}(a)}(\lambda)$ and $\lambda_2 = \chi^-_{\mathrm{ht}(a)}(\lambda)$. Then, ${}^{\langle\nu\rangle}\lambda = {}^{\langle\varnothing\rangle}\lambda_1 \cup {}^{\langle\nu\rangle}\lambda_2$ is a division into blocks in which the first term is trivially marked, and in the second, λ_2 has fewer parts than λ.

If a is a marked part, let b be the second marked part. This time we take $\lambda_1 = \chi^+_{\mathrm{ht}(b)}(\lambda)$ and $\lambda_2 = \chi^-_{\mathrm{ht}(b)}(\lambda)$. This time, ${}^{\langle\nu\rangle}\lambda = {}^{\langle[a>b]\rangle}\lambda_1 \cup {}^{\langle\nu\setminus\{a,b\}\rangle}\lambda_2$ is a division into blocks whose first term is a basic block. Similar arguments work in types C and D. □

Henceforth, all our arguments regarding properties of $\bar{d}$ will address only basic and trivially marked blocks.

Proposition 4.12. *If* ${}^{\langle\nu\rangle}\lambda$ *is a basic block, then* $d_S({}^{\langle\nu\rangle}\lambda)$ *can be computed by the following simplified formulas:*

$$\begin{aligned}\textit{Type B:}\quad & d_S({}^{\langle\nu\rangle}\lambda) = \lambda^{-*}{}_C\\ \textit{Type C:}\quad & d_S({}^{\langle\nu\rangle}\lambda) = \lambda^{+*}{}_B\\ \textit{Type D:}\quad & d_S({}^{\langle\nu\rangle}\lambda) = \lambda^{+-*}{}_D\end{aligned}$$

Proof. If ${}^{\langle\nu\rangle}\lambda$ is a basic block, write $\nu = [n_2 > n_1]$. Let us assume for the time being that ${}^{\langle\nu\rangle}\lambda$ is an ultrabasic block. This will make our calculations less cumbersome. We will obtain a formula; then, at the end of the proof, we use Lemma 4.4 to see that the same formula holds for general basic blocks.

Suppose we are working in type B, so $n_1 = 1$. Let $h_i = \mathrm{ht}_\lambda(n_i)$ for $i = 1, 2$. Thus h_1 is the total number of parts of λ. Let $\mu_1 = \chi^+_{h_2-1}(\lambda)$ and $\mu_2 = \chi^-_{h_2-1}(\lambda)$. Note that μ_1 has only parts of even height, and that $r_{\mu_2}(n_2) = 1$. Let μ_2' be the partition gotten from μ_2 by decreasing the multiplicities of n_2 and n_1 each by 1. Since $n_1 = 1$, we have $\mu_2 = [n_2] \cup \mu'_{2+}$. Writing ${}^{\langle\nu\rangle}\lambda$ as a pair (ν, η), we have $\eta = \mu_1 \cup \mu_2'$, and $\eta^- = \mu_1 \cup \mu_2'^-$. Using Lemma 3.1, we get $\eta^-{}_C = \mu_{1C} \cup \mu_2'^-{}_C$, but since all parts of μ_1 have even height, they all have even multiplicity, so $\mu_{1C} = \mu_1$:

$$\eta^-{}_C = \mu_1 \cup \mu_2'^-{}_C. \tag{14}$$

Next, again using that $n_1 = 1$, we have $\nu \cup \eta^-{}_C = \mu_1 \cup [n_2] \cup \mu_2'^-{}_{C+}$, or

$$(\nu \cup \eta^-{}_C)^* = \mu_1^* \vee [n_2]^* \vee \mu_2'^-{}_C{}^{*+}. \tag{15}$$

We use Lemma 3.1 to get $(\nu \cup \eta^-{}_C)^*{}_C = \mu^*_{1C} \vee ([n_2]^* \vee \mu_2'^-{}_C{}^{*+})_C$. Since μ_1 only has parts of even height, μ_1^* only has even parts, so the C-collapse does nothing. Using Lemma 3.1 yet again, we find that the second term is equal to $[n_2]^* \vee \mu_2'^-{}_C{}^{*+}{}_B$. Now μ_2' is a B-partition, so Spaltenstein's formulas give us that $\mu_2'^-{}_C{}^{*+}{}_B = \mu_2'^*{}_B$. Finally, using the fact that $[n_2]^* \vee \mu_2'^* = \mu_2^{-*}$, we obtain $[n_2]^* \vee \mu_2'^*{}_B = \mu_2^{-*}{}_C$:

$$(\nu \cup \eta^-{}_C)^*{}_C = \mu_1^* \vee \mu_2^{-*}{}_C. \tag{16}$$

Now, we know $\lambda^{-*} = \mu_1^* \vee \mu_2^{-*}$, and Lemma 3.1 would tell us that $\lambda^{-*}{}_C = \mu^*_{1C} \vee \mu_2^{-*}{}_C$. But since μ_1 only has parts of even height, μ_1^* only has even parts, and the C-collapse does nothing do it. Thus (16) is given by $\lambda^{-*}{}_C$, as desired.

We do not give the details in types C and D, but as an aid to those who wish to work them out, we list the analogues of (14), (15), and (16) here.

Type C	*Type D*
$[n_2] \cup \eta^+{}_B = \mu_1 \cup \mu_2^+{}_{B-}$	$\eta^*{}_D{}^* = \mu_1^+ \cup \mu_2'^{*-}{}_C{}^*$
$(\nu \cup \eta^+{}_B)^* = \mu_1^* \vee \mu_2^+{}_B{}^{*-}$	$(\nu \cup \eta^*{}_D{}^*)^* = \mu_1^{+*} \vee [n_2]^* \vee \mu_2'^{*-}{}_C{}^+$
$(\nu \cup \eta^+{}_B)^*{}_B = \mu_1^{+*} \vee \mu^*_{2C}$	$(\nu \cup \eta^*{}_D{}^*)^*{}_D = \mu_1^{+*} \vee \mu_2^{-*}{}_C$

In type C, it turns out to be more convenient not to work with μ_2'. In type D, we need to make use of the identity $\eta^*{}_D{}^* = \eta^{+-}{}_C$. With these points in mind, the proofs are straightforward. □

5. Proofs of the main theorems in the classical groups

In this section, we establish the main theorems of the paper for the classical groups. Theorem 1 is relatively easy: we prove it first, and we make use of it from time to time as we go about proving Theorem 2. The proof of the latter is broken up into a number of steps and occupies most of the section. The steps may look familiar: we end up proving that $\bar{d}$ has many of the properties established in Section 2 before we show that it is actually an extended duality map.

5.1. **The partial order in the classical groups.** The strategy for the proof of the theorem below is quite simple: we just attempt the raw computation of the two values of d_S, using the techniques from the previous section. Those techniques make it straightforward to find a difference in the answers, starting with a difference in the original marking partitions.

Theorem 5.1. *Let $C, C' \subset A(\mathcal{O})$ be two conjugacy classes associated to the same orbit. Then, in the classical groups, $d_{\mathrm{S}}(\mathcal{O}, C) = d_{\mathrm{S}}(\mathcal{O}, C')$ if and only if C and C' have the same image in $\bar{A}(\mathcal{O})$. As a consequence, the partial order (1) is well-defined.*

Proof. We need to prove that if C and C' are two different conjugacy classes in $\bar{A}(\mathcal{O})$, then $d_{\mathrm{S}}(\mathcal{O}, C) \neq d_{\mathrm{S}}(\mathcal{O}, C')$. Suppose that these conjugacy classes are labelled by ${}^{\langle\nu\rangle}\lambda$ and ${}^{\langle\nu'\rangle}\lambda$, respectively. Let a be the largest part of λ that appears in only one of ν and ν'. Therefore, a has (generalized) heights of opposite parity in ν and ν'; assume it has even height in ν. That means that we can break ${}^{\langle\nu\rangle}\lambda$ up into blocks ${}^{\langle\nu_1\rangle}\lambda_1 \cup {}^{\langle\nu_2\rangle}\lambda_2$, where the smallest part of λ_1 is a. (Note that because a is markable, this is a legitimate division into blocks in whatever type we are working in.) But in ${}^{\langle\nu'\rangle}\lambda$, there is some basic block ${}^{\langle\omega\rangle}\zeta$, $\omega = [w_2 > w_1]$, such that $w_2 \geq a > w_1$. We build a division into blocks around this basic block, writing ${}^{\langle\nu'\rangle}\lambda = {}^{\langle\nu'_1\rangle}\lambda'_1 \cup {}^{\langle\omega\rangle}\zeta \cup {}^{\langle\nu'_2\rangle}\lambda'_2$. Finally, let $h = \mathrm{ht}_\lambda(a)$.

Let $\mu = d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda)$, and $\mu' = d_{\mathrm{S}}({}^{\langle\nu'\rangle}\lambda)$. Using Proposition 4.9 just to compute d_{S}, we have

$$\mu^* = d_{\mathrm{S}}({}^{\langle\nu_1\rangle}\lambda_1)^* \cup d_{\mathrm{S}}({}^{\langle\nu_2\rangle}\lambda_2)^*.$$

We see that $\sigma_h(\mu^*) = |d_{\mathrm{S}}({}^{\langle\nu_1\rangle}\lambda_1)^*| = |\lambda_1| = \sigma_h(\lambda)$ in types C or D, and $\sigma_h(\mu^*) = |\lambda_1| - 1 = \sigma_h(\lambda) - 1$ in type B. (This comes from just counting the "$+$" and "$-$" operations that are done in computing d_{S} in each type.)

We now analyze μ'. Write $\zeta = \zeta' \cup \zeta''$, where $\zeta' = \chi^+_{\mathrm{ht}(a)}(\zeta)$ and $\zeta'' = \chi^-_{\mathrm{ht}(a)}(\zeta)$. Suppose we are working in type B; and suppose further that ${}^{\langle\nu'_1\rangle}\lambda'_1$ is nontrivial, so that ${}^{\langle\omega\rangle}\zeta$ is of type D. (Definition 4.8 says basic B-blocks can only occur at the beginning of a marked B-partition.) Then Proposition 4.12 says $d_{\mathrm{S}}({}^{\langle\omega\rangle}\zeta) = \zeta^{+-*}{}_D$. We have $\zeta^{+-*} = \zeta'^{+*} \vee \zeta''^{-*}$, so by Lemma 3.1, $\zeta^{+-*}{}_D = \zeta'^{+*+}{}_{D-} \vee \zeta''^{-*}{}_B$, so

$$\mu'^* = d_{\mathrm{S}}({}^{\langle\nu'_1\rangle}\lambda'_1) \cup \zeta'^{+*+}{}_{D-} \cup \zeta''^{-*}{}_B \cup \bar{d}({}^{\langle\nu'_2\rangle}\lambda'_2).$$

We see that $\sigma_h(\mu'^*) = |d_{\mathrm{S}}({}^{\langle\nu'_1\rangle}\lambda'_1)| + |\zeta'^{+*+}{}_{D-}| = (|\lambda'_1| - 1) + (|\zeta'| + 1) = |\lambda'_1| + |\zeta'| = \sigma_h(\lambda)$. A nearly identical argument establishes that $\sigma_h(\mu'^*) = \sigma_h(\lambda)$ when ${}^{\langle\nu'_1\rangle}\lambda'_1$ is trivial and ${}^{\langle\omega\rangle}\zeta$ is of type B.

Similar computations show that in types C and D, we get $\sigma_h(\mu'^*) = \sigma_h(\lambda) + 1$. Thus, in every case, we get $\sigma_h(\mu'^*) = \sigma_h(\mu^*) + 1$, so $\mu \neq \mu'$, as desired. □

5.2. **Special marked partitions.** The remainder of the section is devoted to establishing Theorem 2 for the classical groups. We begin our attack on it by attempting to characterize the marked partitions that occur in the image of $\bar{d}$. In this subsection, we define the set $\tilde{\mathcal{P}}^{\mathrm{sp}}_X(n)$ of special marked partitions, and show that the image of $\bar{d}$ is contained with this set. Of course, we still have to prove various properties of $\bar{d}$ before we can know that this terminology coincides with the idea of "special" that we introduced in Section 2.

Definition 5.2. Let ${}^{\langle\nu\rangle}\lambda$ be a reduced marked partition, with $\nu = [n_l > \cdots > n_1]$. Assume that l is even, if necessary by taking $n_1 = 0$ in type C. In type B (resp. C, D), ${}^{\langle\nu\rangle}\lambda$ is called *special* if there are no even (resp. odd, even) parts of odd (resp. even, even) height between n_{2i} and n_{2i-1} for $i = 1, \ldots, l/2$; that is, if there are no even (resp. odd, even) parts of odd (resp. even, even) height whose (generalized) height in ν is odd. The set of special marked partitions in $\tilde{\mathcal{P}}_X(n)$ is denoted $\tilde{\mathcal{P}}^{\mathrm{sp}}_X(n)$.

Note that any trivially marked partition is special by this definition, as we expect from Proposition 2.7. On the other hand, if λ is a special partition, a nontrivially marked partition ${}^{\langle\nu\rangle}\lambda$ may be either special or nonspecial.

Lemma 5.3. *We have that $\bar{d}({}^{\langle\varnothing\rangle}\lambda)$ is a special marked partition for any λ.*

Proof. Let us consider the situation in type B. Recalling Lemma 3.3, we can write $d_{\mathrm{S}}({}^{\langle\varnothing\rangle}\lambda) = d_{\mathrm{BV}}(\lambda) = \lambda^{*-}{}_C$. Let π be the list of even parts with odd multiplicity in λ^*. In $\lambda^{*-}{}_C$, some parts of π have odd multiplicity, and others have even multiplicity. According to Lemma 3.2, we could replace π by the set π' obtained by taking the complementary set of even parts with odd multiplicity, together with the same set of even parts with even multiplicity as π. That is, π' is the list of even parts that have even multiplicity in $\lambda^{*-}{}_C$ and odd multiplicity in λ^*, or odd multiplicity in $\lambda^{*-}{}_C$ and even multiplicity in λ^*. We have $d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\pi\rangle}\lambda^{*-}{}_C = {}^{\langle\pi'\rangle}\lambda^{*-}{}_C$; we work with π' for the rest of this proof.

What happens as we pass from λ^{*-} to $\lambda^{*-}{}_C$? We have to make a change in the partition every time we encounter an odd part with odd multiplicity. There are an even number of odd parts with odd multiplicity; we consider them in pairs. Indeed, suppose

$$a_1^{k_1},\, a_2^{k_2},\, \ldots,\, a_l^{k_l} \tag{17}$$

is a list of consecutive odd parts of λ^{*-}, with a_1 and a_l being, say, the largest two odd parts with odd multiplicity. (We are not requiring that a_i and a_{i+1} be consecutive in λ^{*-}, but merely that any parts between them be even.) We have assumed that the multiplicities k_1 and k_l are odd, while $k_2, \ldots, k_{l-1}$ are even. Then, the C-collapse replaces the above parts by the following ones:

$$a_1^{k_1-1}, a_1-1,\, a_2+1, a_2^{k_2-2}, a_2-1,\, \ldots,\, a_l+1, a_l^{k_l-1}. \tag{18}$$

On each such pair of odd parts with odd multiplicity, the C-collapse follows the pattern of the change from (17) to (18); we just investigate what happens on one instance of the pattern. Listing the even parts that have changed multiplicities, we obtain

$$\pi' = [a_1-1, a_2+1, a_2-1, \ldots, a_l+1]. \tag{19}$$

Note that because λ^{*-} has no odd parts between a_{i-1} and a_i, there are no odd parts between $a_{i-1}-1$ and a_i+1 in $\lambda^{*-}{}_C$.

Now, the key observation here is that $\lambda^{*-}{}_C$ is a special C-partition (it equals $d_{\mathrm{BV}}(\lambda)$). Odd parts in special C-partitions have even height and even multiplicity, so any part immediately greater than an odd part also has even height. In particular, each a_i+1 has even height, and is therefore markable. When we pass to the reduced label, for each i, we either retain both parts $a_{i-1}-1$ and a_i+1 (if $a_{i-1}-1$ has even height), or eliminate both of them (if $a_{i-1}-1$ has odd height). Since there are no odd parts between these two parts for any i, we have a special marked partition. □

Proposition 5.4. *We have that $\bar{d}({}^{\langle\nu\rangle}\lambda)$ is a special marked partition for any ${}^{\langle\nu\rangle}\lambda$.*

Proof. The previous lemma establishes this fact for trivially marked blocks, so now we need only consider basic blocks. This is easy to deduce from the formulas given in Proposition 4.12; we work it out in type B now. Let ${}^{\langle\nu\rangle}\lambda$ be a type-B basic block with $\nu = [n_2 > n_1]$, and let $h_1 = \mathrm{ht}_\lambda(n_1)$: we have that h_1 is odd. We can write $\lambda = [1^{h_1}] \vee \lambda'$, where λ' is a C-partition (since λ is a B-partition). Then, $\lambda^{-*}{}_C = ([h_1-1] \cup \lambda'^*)_C$. (Note that $[h_1-1]$ is probably not superior to λ'^*.) Now, h_1-1 is even, so it is unaffected by the C-collapse: $\lambda^{-*}{}_C = [h_1-1] \cup \lambda'^*{}_C$. Since λ' is a C-partition, $\lambda'^*{}_C$ is a special C-partition, in which all odd parts have even height. We claim that the part $[h_1-1]$ "pushes them down" so that they have odd height. Indeed, the only part of of $\lambda'^*{}_C$ larger than h_1-1 is h_1. That is, to be sure, an odd part with even height (which is equal to its multiplicity, n_2-1), but all other parts of $\lambda'^*{}_C$ have their heights increased by 1 when we pass to $\lambda^{-*}{}_C$. But since h_1 is the largest part of $\lambda^{-*}{}_C$, it obviously cannot have odd generalized height in the marking partition. All other odd parts have odd height, so $\bar{d}({}^{\langle\nu\rangle}\lambda)$ is special. □

5.3. **Involutiveness.** Next, we undertake the task of showing that $\bar{d}$ is an involution on the set of special marked partitions. We do this in several stages, beginning just with the trivial conjugacy class on special orbits, then working up to the trivial conjugacy class for all orbits, and finally to the full special set.

Lemma 5.5. *If λ is a special partition, then $\bar{d}({}^{\langle\varnothing\rangle}\lambda) = {}^{\langle\varnothing\rangle}d_{\mathrm{BV}}(\lambda)$.*

Proof. If λ is a special B- (resp. C-, D-) partition, then λ^* is a B- (resp. C-, C-) partition, so all its even (resp. odd, odd) parts have even multiplicity. Therefore, the partition π defined in (8) is trivial. It follows that ρ in (9) is trivial as well, as is the marking partition of $\bar{d}({}^{\langle\nu\rangle}\lambda)$. The underlying partition is then given by $d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda) = d_{\mathrm{BV}}(\lambda)$. □

Lemma 5.6. *For any partition λ, we have that $\bar{d}^2({}^{\langle\varnothing\rangle}\lambda) = {}^{\langle\varnothing\rangle}\lambda$.*

Proof. We know that $\bar{d}({}^{\langle\varnothing\rangle}\lambda)$ is Sommers' canonical inverse for λ, so that $d_{\mathrm{S}}(\bar{d}({}^{\langle\varnothing\rangle}\lambda)) = \lambda$. We therefore have $\bar{d}^2({}^{\langle\varnothing\rangle}\lambda) = {}^{\langle\nu\rangle}\lambda$ for some marking partition ν. We only need to show that $\nu = \varnothing$. To do this, we use Proposition 4.9 to decompose λ into pieces as simple as possible. Call a partition of the form $[a^l]$, with l even, a *rectangle*, and call a partition of the form

$$[a_1^{k_1}, a_2^{k_2}, \ldots, a_m^{k_m}], \tag{20}$$

with k_1 and k_m odd and $k_2, \ldots, k_{m-1}$ even, a *staircase*. Any D-partition can be written as a union of rectangles and staircases. Let us define a *partial staircase* to be a staircase from which either $a_1^{k_1}$ or $a_m^{k_m}$ is omitted: a *lower partial staircase* in the former case, and an *upper partial staircase* in the latter. In seeking a division into blocks, we can write any B-partition as a lower partial staircase followed by some number of rectangles and staircases, and any C-partition as a union of rectangles and staircases, possibly followed by an upper partial staircase. The proof of this lemma is accomplished by proving it separately for each of these kinds of blocks.

In many cases, showing that ν is trivial is easy because λ just does not have any possible marking partitions. Rectangles and staircases have only one part of even height, while upper partial staircases have none; and lower partial staircases have only one part of odd height. The lemma follows in completely when $\bar{d}({}^{\langle\varnothing\rangle}\lambda)$ is of type B or D (where marking partitions must have an even number of parts), and for upper partial staircases in type C. For rectangles in type C, the statement is a consequence of Lemma 4.4. The only remaining case is that of staircases in type C, which we treat now.

Let λ be a C-partition of the form (20), and let $\mu = d_{\mathrm{S}}({}^{\langle\varnothing\rangle}\lambda)$. We have $\mu = \lambda^{+}{}_B{}^*$; moreover, we claim that $\bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\varnothing\rangle}\mu$. Note that in λ, a_1 and a_m must be even. All the parts $a_1, \ldots, a_{m-1}$ have odd height; only a_m has even height. Therefore, λ^* has only one even part, its largest one, and the multiplicity of that part is a_m, which is even. Therefore, π as defined in (8) is trivial, so $\bar{d}({}^{\langle\nu\rangle}\lambda)$ is trivially marked.

We could just trudge ahead and compute $\bar{d}({}^{\langle\varnothing\rangle}\mu)$ directly, but instead, we use the following trick. We have already observed that the proposition holds for B-partitions, so $\bar{d}^2({}^{\langle\varnothing\rangle}\mu) = {}^{\langle\varnothing\rangle}\mu$. That is, $\bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\varnothing\rangle}\mu$. But we also have $\bar{d}({}^{\langle\varnothing\rangle}\lambda) = {}^{\langle\varnothing\rangle}\mu$, so if we had $\nu \neq \varnothing$, that would contradict Theorem 5.1. Therefore, $\nu = \varnothing$, and proposition holds for C-staircases. $\square$

Lemma 5.7. *Let ${}^{\langle\nu\rangle}\lambda$ be a special ultrabasic block, with $\nu = [n_2 > n_1]$. If ${}^{\langle\nu\rangle}\lambda$ is of type B (resp. C, D), let m be the largest part of even (resp. odd, even) height in $d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda)$. (When ${}^{\langle\nu\rangle}\lambda$ is of type B, we put $m = 0$ if $d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda)$ has no parts of even height.) If $m > 1$, then $\bar{d}({}^{\langle\nu\rangle}\lambda)$ is again a special ultrabasic block, given by the following formulas:*

$$\begin{aligned} &\textit{Type } B: \quad \bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle[m]\rangle}\lambda^{-*}, \\ &\textit{Type } C: \quad \bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle[m,1]\rangle}\lambda^{+*}, \\ &\textit{Type } D: \quad \bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle[m,1]\rangle}\lambda^{+-*}. \end{aligned}$$

If $m \leq 1$, then $\bar{d}({}^{\langle\nu\rangle}\lambda)$ is a trivially marked partition, whose underlying partition is as given above.

Proof. To prove this, we must dive into the details of the proof of Proposition 5.4. Suppose ${}^{\langle\nu\rangle}\lambda$ is of type B; recall that we wrote $\lambda = [1^{h_1}] \vee \lambda'$, where λ' is a C-partition. If ${}^{\langle\nu\rangle}\lambda$ is special, then all its even parts must have even heights. (This is true for even parts smaller than n_2 by the definition of "special," and for even parts larger than n_2 by the definition of "basic block.") This means that in λ', all odd parts have even heights; *i.e.*, λ' is a special C-partition. Therefore, $\lambda'^*{}_C = \lambda'^*$, and $\lambda^{-*}{}_C = [h_1 - 1] \cup \lambda'^* = \lambda^{-*}$. (Here we have used the fact that $n_1 = 1$, so h_1 does not occur as a part of λ'^*.)

Now, $\lambda^* = [h_1] \cup \lambda'^*$ does not have any odd parts with odd multiplicity other than h_1, because λ'^* is a C-partition. We have $\lambda^* = \eta^{*+} \vee [1^{n_2}]$, so in η^*, there are no even parts of height less than or equal to n_2 that have odd multiplicity. Thus π as defined in (8) only has parts of height greater than n_2. Now, back in λ, all parts larger than n_2 have even height, so in η^* or λ^*, all parts whose height is greater than n_2 must be even. Let b be the largest part of λ^* that has even height greater than n_2; in other words, b is the largest markable part smaller than h_2. (If there are no markable parts smaller than h_2, take $b = 0$.) There is an odd number of even parts that are greater than or equal to b and smaller than h_2, but an even number of even parts smaller than b and greater than or equal to any smaller markable part. It follows that when we pass to the reduced marked partition to compute ${}^{\langle\rho\rangle}\tau$ in (9), we get $\rho = [b]$.

Finally, we look at $\bar{d}({}^{\langle\nu\rangle}\lambda) = {}^{\langle\hat{\nu}\cup[b]\rangle}\lambda^{-*}$. Since b is the largest markable part smaller than h_2, it is clear that we can replace $[h_1 - 1 > h_2 - 1 \geq b]$ by $[h_1 - 1]$ without changing the reduced marked partition to which it is equivalent. Now, let m be the largest part of even height in λ^{-*}, or, if there are no parts of even height, take $m = 0$. If $m \neq 0$, suppose its height is k. That means that in λ, k is an even part of height m. Since ${}^{\langle\nu\rangle}\lambda$ is special, k must have even height, so m is necessarily even. Therefore, m is markable in λ^{-*}. In the case of either $m \neq 0$ or $m = 0$, then, we see that ${}^{\langle [m] \rangle}\lambda^{-*}$ is the reduced marked partition equivalent to ${}^{\langle [h_1-1] \rangle}\lambda^{-*}$. □

Proposition 5.8. *If ${}^{\langle \nu \rangle}\lambda$ is a special marked partition, then $\bar{d}^2({}^{\langle \nu \rangle}\lambda) = {}^{\langle \nu \rangle}\lambda$.*

Proof. Lemma 5.6 established this fact for trivially marked partitions, so now we only need to consider basic blocks. Indeed, we actually restrict ourselves to ultrabasic blocks, since we can then use Lemma 4.4 to pass up to the result for arbitrary basic blocks. Let ${}^{\langle \nu \rangle}\lambda$ be a special ultrabasic block, and let $\bar{d}({}^{\langle \nu \rangle}\lambda) = {}^{\langle \xi \rangle}\mu$. According to Lemma 5.7, there are two cases to consider: either ${}^{\langle \xi \rangle}\mu$ is trivially marked, or it is again a special ultrabasic block.

First, suppose it is trivially marked. In each type, we can directly compute $d_{\mathrm{S}}({}^{\langle \varnothing \rangle}\mu) = d_{\mathrm{BV}}(\mu)$: in type B, for example, we have $\mu = \lambda^{-*}$, and $d_{\mathrm{BV}}(\mu) = \mu^{+}{}_{B}{}^{*}{}_{B} = \lambda^{*}{}_{B}{}^{*}{}_{B}$. (To get $\mu^+ = \lambda^*$, we had to use the fact that $n_1 = 1$.) Moreover, according to Lemma 5.7, the fact that $\bar{d}({}^{\langle \nu \rangle}\lambda)$ is trivially marked means that μ has no parts of even height, which in turn means that $\mu^* = \lambda^-$ has no even parts. Again using that $n_1 = 1$, it follows that λ has no even parts, and is therefore automatically a special B-partition. We deduce that $\lambda^{*}{}_{B}{}^{*}{}_{B} = \lambda$. Is it possible that $\bar{d}({}^{\langle \varnothing \rangle}\mu) = {}^{\langle \nu' \rangle}\lambda$ for some $\nu' \neq \nu$? Let us again use the trick from the end of the proof of Lemma 5.6. We know from Lemma 5.6 that $\bar{d}^2({}^{\langle \varnothing \rangle}\mu) = {}^{\langle \varnothing \rangle}\mu$, but having $\bar{d}({}^{\langle \nu' \rangle}\lambda) = {}^{\langle \varnothing \rangle}\mu = \bar{d}({}^{\langle \nu \rangle}\lambda)$ for $\nu' \neq \nu$ would contradict Theorem 5.1. Thus, $\bar{d}^2({}^{\langle \nu \rangle}\lambda) = {}^{\langle \nu \rangle}\lambda$.

Now, suppose instead that $\xi \neq \varnothing$. This time, ${}^{\langle \xi \rangle}\mu$ is itself a special ultrabasic block, so we can use the formulas of Lemma 5.7 twice in a row to establish the result. For instance, starting in type B, we have $\mu = \lambda^{-*}$, so $\bar{d}({}^{\langle \xi \rangle}\mu) = {}^{\langle [p,1] \rangle}\mu^{+*}$, where p is the largest part of odd height in μ^{+*}. But $\mu^{+*} = \lambda$, as argued in the previous paragraph, and n_2 is the largest part of odd height in λ. In this case as well, we find that $\bar{d}^2({}^{\langle \nu \rangle}\lambda) = {}^{\langle \nu \rangle}\lambda$. □

5.4. **Specialization.** The final step is to define a map for passing from a given marked partition to special one that is larger than it in the partial order. After we show that that this coincides with $\bar{d}^2$, we will be in a position prove that $\bar{d}$ satisfies the remaining axioms for an extended duality map. We begin with a map which we call the *partial specialization map* $s : \tilde{\mathcal{P}}_\epsilon(n) \to \tilde{\mathcal{P}}_\epsilon(n)$, defined as follows. If ${}^{\langle \nu \rangle}\lambda$ is a nonspecial marked partition of type B (resp. C, D), let a be the smallest even (resp. odd, even) part of odd (resp. even, even) height in λ and odd height in ν. (Of course, no such a exists for a special marked partition). The part a must have even multiplicity, say l. Let λ' be the partition gotten from λ by deleting all l copies of a. We put

$$s({}^{\langle \nu \rangle}\lambda) = \begin{cases} {}^{\langle \nu \rangle}\lambda & \text{if } {}^{\langle \nu \rangle}\lambda \text{ is special,} \\ {}^{\langle \nu \rangle}(\lambda' \cup [a+1, a^{l-2}, a-1]) & \text{if } {}^{\langle \nu \rangle}\lambda \text{ is nonspecial.} \end{cases}$$

Of course, we may have to pass to the reduced marked partition from the above formula, if it happens that $a+1$ was a markable part of λ and was, in fact, marked.

It is clear that for a nonspecial marked partition, the map s decreases the total number of even (resp. odd, even) parts of odd (resp. even, even) height in the underlying partition and odd height in the marking partition. By induction on that quantity, we obtain the following result.

Lemma 5.9. *Given a marked partition ${}^{\langle \nu \rangle}\lambda$, there is some nonnegative integer N such that $s^N({}^{\langle \nu \rangle}\lambda)$ is special.* □

We now define the specialization map $e : \tilde{\mathcal{P}}_\epsilon(n) \to \tilde{\mathcal{P}}^{\mathrm{sp}}_\epsilon(n)$ as

$$e({}^{\langle \nu \rangle}\lambda) = s^N({}^{\langle \nu \rangle}\lambda),$$

where N is taken large enough that the right-hand side is special. Note that since the map s fixes special marked partitions, there is no ambiguity in the above definition arising from the particular choice of N.

Proposition 5.10. *We have that $\bar{d}({}^{\langle \nu \rangle}\lambda) = \bar{d}(s({}^{\langle \nu \rangle}\lambda))$ for any marked partition ${}^{\langle \nu \rangle}\lambda$.*

Proof. We begin by proving that $d_{\mathrm{S}}({}^{\langle \nu \rangle}\lambda) = d_{\mathrm{S}}(s({}^{\langle \nu \rangle}\lambda))$. Let us assume that ${}^{\langle \nu \rangle}\lambda$ is a nonspecial basic block of type B, let a and l be as in the definition of s, and let $h = \mathrm{ht}_\lambda(a)$. Let ${}^{\langle \omega \rangle}\zeta = s({}^{\langle \nu \rangle}\lambda) = (\omega, \kappa)$, and let b be the next smaller part of λ after a. Since a is the smallest even part with odd height, b must either be odd or have even height. But if b has even height, it must have odd multiplicity (since a has odd height), so b is necessarily odd in all cases. Suppose $\nu = [n_2 > n_1]$. If $n_2 > a+1$, then $\omega = \nu$; otherwise, $\omega = [m > n_1]$, where m is the largest odd part of λ that is smaller than a and has odd height.

Now, h is an odd part with even height a in λ^*, and $h-l$ (also odd) is the next smaller part after h. Write $\lambda^{-*} = \tau_1 \cup [h^{a-b}, h-l] \cup \tau_2$, where $\tau_1 = \chi_b^+(\lambda^{-*})$ and $\tau_2 = \chi_{a-1}^-(\lambda^{-*})$. Using Lemma 3.1, it is easy to check that

$$d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda) = {\lambda^{-*}}_C = \tau_{1C} \cup [h^{a-b-1}, h-1, h-l+1] \cup \tau_{2C}.$$

We get ζ from λ by replacing $[a^l]$ by $[a+1, a^{l-2}, a-1]$. Then ζ^* looks like λ^*, except that the portion of the form $[h^{a-b}, h-l]$ has been changed to $[h^{a-b-1}, h-1, h-l+1]$. In the case that $\omega = \nu$, we just compare with the above computation to see that ${\lambda^{-*}}_C = {\zeta^{-*}}_C$; *i.e.*, $d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda) = d_{\mathrm{S}}(s({}^{\langle\nu\rangle}\lambda))$. But even if $\omega \neq \nu$, we recall that it is not necessary to pass to reduced marked partition when computing d_{S} (which is, after all, defined as a map $\mathcal{N}_{\mathrm{o,c}} \to {}^L\mathcal{N}_{\mathrm{o}}$), so we can simply replace ω by ν and apply the above argument anyway.

It remains to verify that $\bar{d}({}^{\langle\nu\rangle}\lambda)$ and $\bar{d}(s({}^{\langle\nu\rangle}\lambda))$ produce the same marking partition. This is straightforward but extremely tedious. The proof consists of writing down the various intermediate marked partitions occurring in (9), while scrupulously remembering to pass to a reduced marked partition whenever possible. The cases of $\omega = \nu$ and $\omega \neq \nu$ must be considered separately; the former is slightly easier. We omit the details. □

Corollary 5.11. *We have that $\bar{d} \circ e = \bar{d}$, and that $\bar{d}^2 = e$.*

Proof. The first statement is an immediate consequence of the preceding proposition, since, by induction, we have $\bar{d} = \bar{d} \circ s^N$ for all $N \geq 0$. Then, on the one hand, we can apply $\bar{d}$ to both sides again to obtain $\bar{d}^2 \circ e = \bar{d}^2$; but on the other hand, we know by Proposition 5.8 that $\bar{d}^2$ is the identity on special marked partitions, and the image of e consists of special marked partitions, so $\bar{d}^2 \circ e = e$. Thus $\bar{d}^2 = e$. □

Proposition 5.12. *We have that $\bar{d}^2({}^{\langle\nu\rangle}\lambda) \geq {}^{\langle\nu\rangle}\lambda$ for any marked partition ${}^{\langle\nu\rangle}\lambda$.*

Proof. It is easy to see, by construction, that the underlying partition of $s({}^{\langle\nu\rangle}\lambda)$ dominates λ. Combining this with Proposition 5.10, we see that $s({}^{\langle\nu\rangle}\lambda) \geq {}^{\langle\nu\rangle}\lambda$. It follows that $e({}^{\langle\nu\rangle}\lambda) \geq {}^{\langle\nu\rangle}\lambda$; *i.e.* that $\bar{d}^2({}^{\langle\nu\rangle}\lambda) \geq {}^{\langle\nu\rangle}\lambda$, as desired. □

Lemma 5.13. *Suppose that ${}^{\langle\nu\rangle}\lambda \leq {}^{\langle\nu'\rangle}\lambda'$, and that ${}^{\langle\nu'\rangle}\lambda'$ is special. Then $s({}^{\langle\nu\rangle}\lambda) \leq {}^{\langle\nu'\rangle}\lambda'$ as well.*

Proof. The argument used to prove this statement is similar in flavor to the argument we gave for Theorem 5.1. Assume that ${}^{\langle\nu\rangle}\lambda$ is nonspecial. Let $\mu = d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda)$ and $\mu' = d_{\mathrm{S}}({}^{\langle\nu'\rangle}\lambda')$. We have that $\lambda \leq \lambda'$ and $\mu \geq \mu'$. Since $d_{\mathrm{S}}(s({}^{\langle\nu\rangle}\lambda)) = d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda)$, all we have to prove is that the underlying partition of $s({}^{\langle\nu\rangle}\lambda)$ is smaller than λ'. Let ζ denote the underlying partition of $s({}^{\langle\nu\rangle}\lambda)$. Let a, l, and h be as in the proof of Proposition 5.10. A brief consideration of how ζ is formed reveals the following relationship between ζ and λ:

$$\begin{aligned} \sigma_{h-l+i}(\zeta) &= \sigma_{h-l+i}(\lambda) + 1 && \text{for } i = 1, \dots, l-1 \\ \sigma_k(\zeta) &= \sigma_k(\lambda) && \text{for } k \neq h-l+1, \dots, h-1 \end{aligned}$$

We know $\sigma_k(\lambda) \leq \sigma_k(\lambda')$ for all k, but to establish $\zeta \leq \lambda'$, we need to prove the following stronger statements:

$$\sigma_{h-l+i}(\lambda) + 1 \leq \sigma_{h-l+i}(\lambda') \qquad \text{for } i = 1, \dots, l-1 \tag{21}$$

Let us assume that the above fails for some i; we shall derive a contradiction. Suppose, in particular, that it fails for $i = j$. This means that $\sigma_{h-l+j}(\lambda) = \sigma_{h-l+j}(\lambda')$. Let $b_1 \geq \cdots \geq b_l$ be the $(h-l+1)$-th, …, h-th parts of λ', respectively. We have

$$\sigma_{h-l+j+1}(\lambda) = \sigma_{h-l+j}(\lambda) + a \leq \sigma_{h-l+j+1}(\lambda') = \sigma_{h-l+j}(\lambda') + b_{j+1}, \tag{22}$$

so $a \leq b_{j+1}$. We also have

$$\sigma_{h-l+j-1}(\lambda) = \sigma_{h-l+j}(\lambda) - a \leq \sigma_{h-l+j-1}(\lambda') = \sigma_{h-l+j}(\lambda') - b_j, \tag{23}$$

which implies $a \geq b_j$. Since $b_j \geq b_{j+1} \geq a$, we conclude that $a = b_j = b_{j+1}$. But then (22) says that $\sigma_{h-l+j+1}(\lambda) = \sigma_{h-l+j+1}(\lambda')$, so (21) fails for $i = j+1$ as well. If i_0 is the smallest value of i for which (21) fails, we see by induction that it fails for $i = i_0 + 1, \dots, l$ as well. Furthermore, $b_{i_0} = \cdots = b_l = a$.

We claim, moreover, that $i_0 = 1$; *i.e.* that (21) fails for all i. If not, the inequality (23) can be strengthened using the fact that the (i_0-1)-th inequality in (21) holds:

$$\sigma_{h-l+i_0-1}(\lambda) + 1 = \sigma_{h-l+i_0}(\lambda) - a + 1 \leq \sigma_{h-l+i_0-1}(\lambda') = \sigma_{h-l+i_0}(\lambda') - b_{i_0}. \tag{24}$$

We deduce that $a - 1 \geq b_{i_0}$. Since $b_{i_0} \geq b_{i_0+1} \geq a$, we obtain $a - 1 \geq a$, a contradiction. We thus have $\sigma_{h-l+i}(\lambda) = \sigma_{h-l+i}(\lambda')$ and $b_i = a$ for $i = 1, \ldots, l$. Additionally, (23) also gives us that $\sigma_{h-l}(\lambda) = \sigma_{h-l}(\lambda')$.

We claim that a must have odd height in λ' if we are in type B, and even height in types C and D. We prove it in type B as follows. The first $h-l$ parts of λ constitute a B-partition, so $\sigma_{h-l}(\lambda) = \sigma_{h-l}(\lambda')$ is odd. Suppose a had even height in λ', and let k be the height of the next larger part of λ'. We know that k must be even too, since a must have even multiplicity. Then $\sigma_{h-l}(\lambda') = \sigma_k(\lambda') + (h - l - k)a$. Since the second term here is even, $\sigma_k(\lambda')$ must be odd. But since k is even, the first k parts of λ' constitute a D-partition, and $\sigma_k(\lambda')$ has to be even. We have a contradiction; therefore, a has odd height in λ'.

Since ${}^{\langle\nu'\rangle}\lambda'$ is special, a must have even height with respect to ν'; it cannot appear inside a basic block. We continue to take k to be the height of the next larger part of λ' after a, but we know now that k is odd. Let $\theta' = \chi_k^+(\lambda')$, and let κ' be the partition consisting of those parts of ν' that are larger than a. Then ${}^{\langle\kappa'\rangle}\theta'$ is a marked B-partition, and if we let $m = r_{\lambda'}(a)$, then the expression ${}^{\langle\kappa'\rangle}\theta' \cup {}^{\langle\varnothing\rangle}[a^m] \cup \cdots$ is part of a division into blocks of ${}^{\langle\nu'\rangle}\lambda'$. We can compute, then, that

$$\mu'^* = d_{\mathrm{S}}({}^{\langle\kappa'\rangle}\theta')^* \cup [a^m] \cup \cdots .$$

Since ${}^{\langle\kappa'\rangle}\theta'$ is of type B, we compute that

$$\sigma_{h-l}(\mu'^*) = |\theta'| - 1 + (h - l - k)a = \sigma_{h-l}(\lambda') - 1.$$

On the other hand, in ${}^{\langle\nu\rangle}\lambda$, the part a belongs to some nonspecial basic block ${}^{\langle\gamma\rangle}\phi$, around which we can build a division into blocks ${}^{\langle\kappa\rangle}\theta \cup {}^{\langle\gamma\rangle}\phi \cup \cdots$. We either have that ${}^{\langle\kappa\rangle}\theta$ is of type B and ${}^{\langle\kappa\rangle}\phi$ of type D, or that ${}^{\langle\kappa\rangle}\theta$ is trivial and ${}^{\langle\kappa\rangle}\phi$ is of type B. Assume we are in the former case; the latter is handled similarly. Let $h' = \mathrm{ht}_\phi(a)$, and let $\phi' = \chi^+_{h'-l}(\phi)$ and $\phi'' = \chi^-_{h'-l}(\phi)$. Using Proposition 4.12 and Lemma 3.1 to compute $d_{\mathrm{S}}({}^{\langle\gamma\rangle}\phi)$, we fine

$$\mu^* = d_{\mathrm{S}}({}^{\langle\kappa\rangle}\theta)^* \cup \phi'^{+*+}{}_{D-}{}^* \cup \phi''^{-*}{}_B{}^* \cup \cdots .$$

We obtain

$$\sigma_{h-l}(\mu^*) = (|\theta'| - 1) + (|\phi'| + 1) = \sigma_{h-l}(\lambda)$$

In particular, we see that $\sigma_{h-l}(\mu^*) \not\leq \sigma_{h-l}(\mu'^*)$, which contradicts the assumption that $\mu^* \leq \mu'^*$. Therefore, the inequalities (21) hold for all i, and we obtain $\zeta \leq \lambda'$, as desired. □

Proposition 5.14. *If* ${}^{\langle\nu\rangle}\lambda \leq {}^{\langle\nu'\rangle}\lambda'$*, then* $\bar{d}({}^{\langle\nu\rangle}\lambda) \geq \bar{d}({}^{\langle\nu'\rangle}\lambda')$.

Proof. We first prove the statement in the special case that ${}^{\langle\nu\rangle}\lambda$ and ${}^{\langle\nu'\rangle}\lambda'$ are special. The $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$-inequality $\bar{d}({}^{\langle\nu\rangle}\lambda) \geq \bar{d}({}^{\langle\nu'\rangle}\lambda')$ is equivalent to the two $\mathcal{N}_{\mathrm{o}}$-inequalities

$$d_{\mathrm{S}}({}^{\langle\nu\rangle}\lambda) \geq d_{\mathrm{S}}({}^{\langle\nu'\rangle}\lambda') \qquad \text{and} \qquad d_{\mathrm{S}}(\bar{d}({}^{\langle\nu\rangle}\lambda)) \leq d_{\mathrm{S}}(\bar{d}({}^{\langle\nu'\rangle}\lambda')). \tag{25}$$

The first of these is implied by ${}^{\langle\nu\rangle}\lambda \leq {}^{\langle\nu'\rangle}\lambda'$, by definition. For the second, since these marked partitions are special, we know $d_{\mathrm{S}}(\bar{d}({}^{\langle\nu\rangle}\lambda)) = \lambda$ and $d_{\mathrm{S}}(\bar{d}({}^{\langle\nu'\rangle}\lambda')) = \lambda'$. But the inequality $\lambda \leq \lambda'$ is again part of the definition of ${}^{\langle\nu\rangle}\lambda \leq {}^{\langle\nu'\rangle}\lambda'$. Thus, (25) holds, and the proposition holds for special marked partitions.

Now, if ${}^{\langle\nu\rangle}\lambda$ and ${}^{\langle\nu'\rangle}\lambda'$ are arbitrary marked partitions with ${}^{\langle\nu\rangle}\lambda \leq {}^{\langle\nu'\rangle}\lambda'$, we obtain ${}^{\langle\nu\rangle}\lambda \leq \bar{d}^2({}^{\langle\nu'\rangle}\lambda')$ by Proposition 5.12. Then, repeated application of Lemma 5.13 implies that $e({}^{\langle\nu\rangle}\lambda) = \bar{d}^2({}^{\langle\nu\rangle}\lambda) \leq \bar{d}^2({}^{\langle\nu'\rangle}\lambda')$. Both sides of this inequality are special marked partitions, so the previous paragraph tells us that $\bar{d}^3({}^{\langle\nu\rangle}\lambda) \geq \bar{d}^3({}^{\langle\nu'\rangle}\lambda')$. Finally, Proposition 5.8, combined with Proposition 5.4, says that $\bar{d}^3 = \bar{d}$, so we get $\bar{d}({}^{\langle\nu\rangle}\lambda) \geq \bar{d}({}^{\langle\nu\rangle}\lambda)$, as desired. □

We have now established that $\bar{d}$ satisfies each of the axioms (1), (2), (3), and (4), in Propositions 5.14, 5.12, 4.2, and 4.3, respectively. We have therefore established the following.

Theorem 5.15. *The map* $\bar{d}$ *is an extended duality map in the classical groups.* □

6. Explicit calculations and the exceptional groups

The main results in the case of the exceptional groups are established by explicit calculation. In this section, we present explicit calculations of the partial order and the duality map in all of the exceptional groups, as well as in a number of classical groups of small rank.

We name elements $(\mathcal{O}, C) \in \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ in the exceptional groups by a pair of symbols (L_1, L_2), where L_1 is the Bala-Carter notation for $\mathcal{O}$, as found in, say, [7], and L_2 is the label Sommers assigns to $(\mathcal{O}, C)$ in his generalized Bala-Carter theorem [15]. (Of course, we are only writing down L_1 for our own convenience, since L_2 alone determines the orbit.) We deviate from this notation when C is the trivial conjugacy class in $\bar{A}(\mathcal{O})$: in this case, the generalized Bala-Carter label for $(\mathcal{O}, C)$ is the same as the Bala-Carter label for $\mathcal{O}$, but for the sake of brevity, we write $(L_1, 1)$ rather than (L_1, L_1).

A further comment about generalized Bala-Carter labels for pairs $(\mathcal{O}, C)$ is in order, because the generalized Bala-Carter theorem is actually a classification of $\mathcal{N}_{\mathrm{o,c}}$, not of $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$. For most orbits in the exceptional groups, we have $A(\mathcal{O}) = \bar{A}(\mathcal{O})$, so this distinction does not matter, but in a handful of cases, $\bar{A}(\mathcal{O})$ has fewer conjugacy classes than $A(\mathcal{O})$. This occurs for two orbits in F_4, two in E_7, and seven in E_8. In all but one of these cases, we have $A(\mathcal{O}) = S_2$ and $\bar{A}(\mathcal{O}) = 1$; however, for the orbit $E_8(b_6)$ in type E_8, we have $A(\mathcal{O}) = S_3$, $\bar{A}(\mathcal{O}) = S_2$. In all of these cases, the only ambiguity is that two conjugacy classes of $A(\mathcal{O})$ map to the trivial conjugacy class of $\bar{A}(\mathcal{O})$. (In the $E_8(b_6)$ example, only one conjugacy class of S_3 goes to the nontrivial conjugacy class of S_2.) In all cases, we simply ignore the nontrivial class of $A(\mathcal{O})$ that maps to the trivial one in $\bar{A}(\mathcal{O})$, and we designate the latter with a label of the form $(L_1, 1)$.

Theorem 6.1. *Let $C, C' \subset A(\mathcal{O})$ be two conjugacy classes associated to the same orbit. Then, in the exceptional groups, $d_{\mathrm{S}}(\mathcal{O}, C) = d_{\mathrm{S}}(\mathcal{O}, C')$ if and only if C and C' have the same image in $\bar{A}(\mathcal{O})$. As a consequence, the partial order (1) is well-defined in the exceptional groups.*

Proof. Sommers gives tables of all the values of d_{S} on all pairs $(\mathcal{O}, C) \in \mathcal{N}_{\mathrm{o,c}}$ for each exceptional group in [16]. We merely read through this table and verify that the above statement is true. ☐

Theorem 6.2. *There exists an extended duality map in the case of each exceptional group. It is unique in types G_2, E_6, and E_7. Furthermore, its value is uniquely determined on all but two conjugacy classes in type F_4, and on all but four conjugacy classes in type E_8. There are four possibilities for $\bar{d}$ in type F_4, given by the following choices:*

$$\begin{aligned} \bar{d}(C_3(a_1), A_1 + B_2) &= (C_3(a_1), 1) \qquad &&\text{or} \quad (C_3(a_1), A_1 + B_2), \\ \bar{d}(B_2, A_3) &= (B_2, 1) &&\text{or} \quad (B_2, A_3). \end{aligned}$$

There are six possibilities for $\bar{d}$ in type E_8, given by the following choices:

$$\begin{aligned} \bar{d}(E_7(a_5), A_5 + A_2) &= (E_6(a_3) + A_1, 1) \quad &&\text{or} \quad (E_6(a_3) + A_1, A_5 + 2A_1), \\ \bar{d}(E_7(a_5), A_1 + D_6(a_2)) &= (E_7(a_5), 1) &&\text{or} \quad (E_7(a_5), A_1 + D_6(a_2)), \\ \bar{d}(D_6(a_2), D_4 + A_3) &= (D_6(a_2), 1) &&\text{or} \quad (D_6(a_2), D_4 + A_3), \end{aligned}$$

with the restriction that if $\bar{d}(E_7(a_5), A_5 + A_2) = (E_6(a_3) + A_1, A_5 + 2A_1)$, then $\bar{d}(E_7(a_5), A_1 + D_6(a_2)) = (E_7(a_5), A_1 + D_6(a_2))$. The value of $\bar{d}(E_6(a_3) + A_1, A_5 + 2A_1)$ is also ambiguous, but it is determined once choices have been made in the three cases above. We have

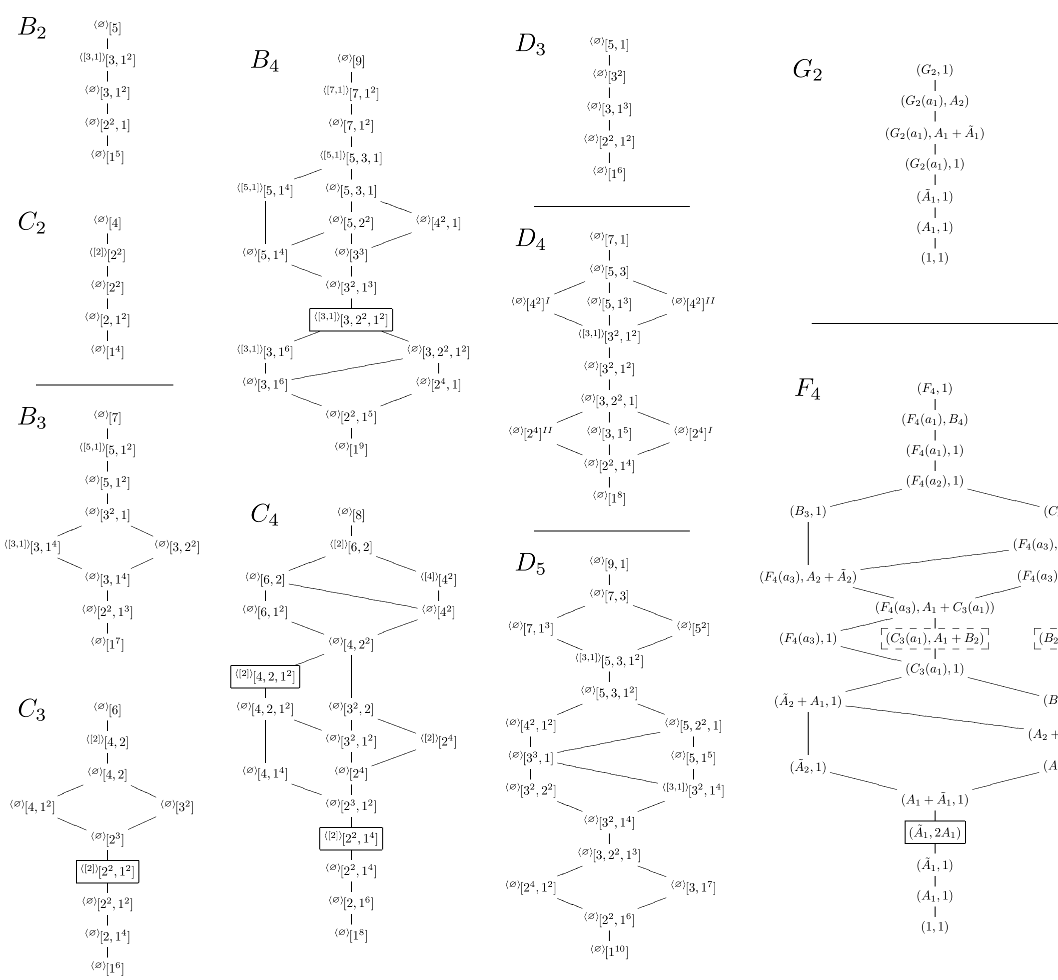
$$\bar{d}(E_6(a_3) + A_1, A_5 + 2A_1) = \begin{cases} (E_7(a_5), A_5 + A_2) & \text{if } \bar{d}(E_7(a_5), A_5 + A_2) = (E_6(a_3) + A_1, A_5 + 2A_1), \\ \bar{d}(E_7(a_5), A_1 + D_6(a_2)) & \text{otherwise.} \end{cases}$$

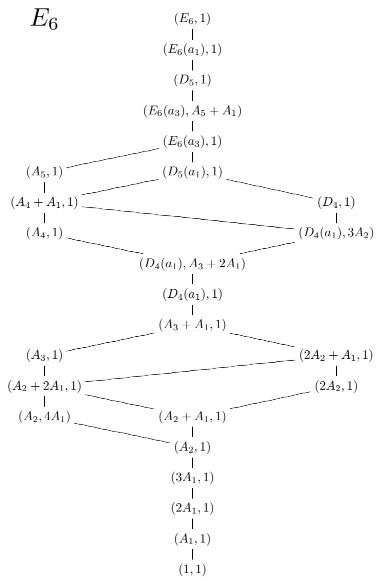
Proof. Once we have drawn out the partial-order diagram of $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ for the exceptional groups, we can attempt to construct an extended duality map by hand, starting with the axioms (3) and (4), regarding agreement with d_{S}. Sometimes those two axioms alone determine a map completely, if, for any $(\mathcal{O}, C) \in \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ with $C \neq 1$, we have that $\bar{A}(d_{\mathrm{S}}(\mathcal{O}, C))$ is trivial. Then axiom (3) determines $\bar{d}$ on such pairs, while (4) determines it on pairs of the form $(\mathcal{O}, 1)$. This situation occurs in types G_2, E_6, and E_7. We then verify by inspection that the map thus defined also satisfies axioms (1) and (2).

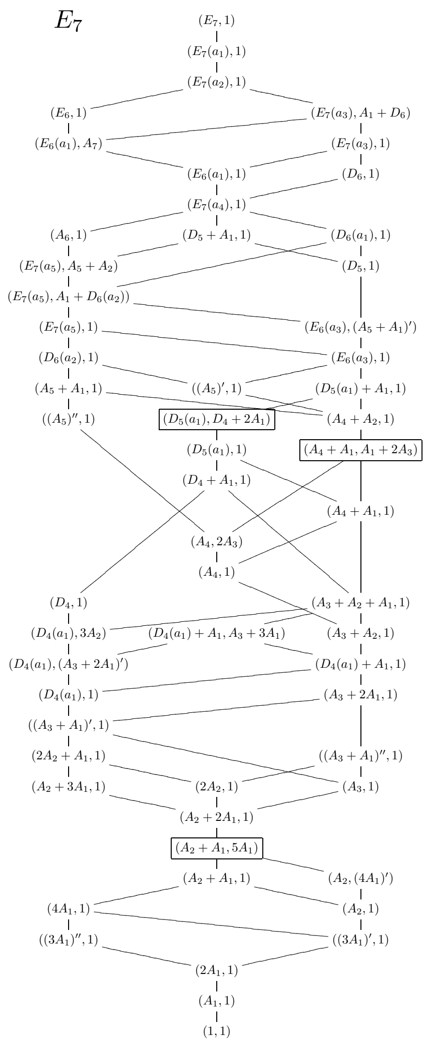
The only instances of pairs $(\mathcal{O}, C)$ with $C \neq 1$ such that $\bar{A}(d_{\mathrm{S}}(\mathcal{O}, C))$ is nontrivial are $(C_3(a_1), A_1 + B_2)$ and (B_2, A_3) in type F_4, and $(E_7(a_5), A_5 + A_2)$, $(E_7(a_5), A_1 + D_6(a_2))$, $(E_6(a_3) + A_1, A_5 + 2A_1)$, and $(D_6(a_2), D_4 + A_3)$ in type E_8. For each of the two such pairs in F_4, we actually have $d_{\mathrm{S}}(\mathcal{O}, C) = \mathcal{O}$, and that $\bar{A}(\mathcal{O}) \simeq S_2$. There are thus two possibilities for each $\bar{d}(\mathcal{O}, C)$, as recorded above. We must then verify that each of the four possibilities actually satisfies axioms (1) and (2) as well. That verification is quite easy, and we omit the details.

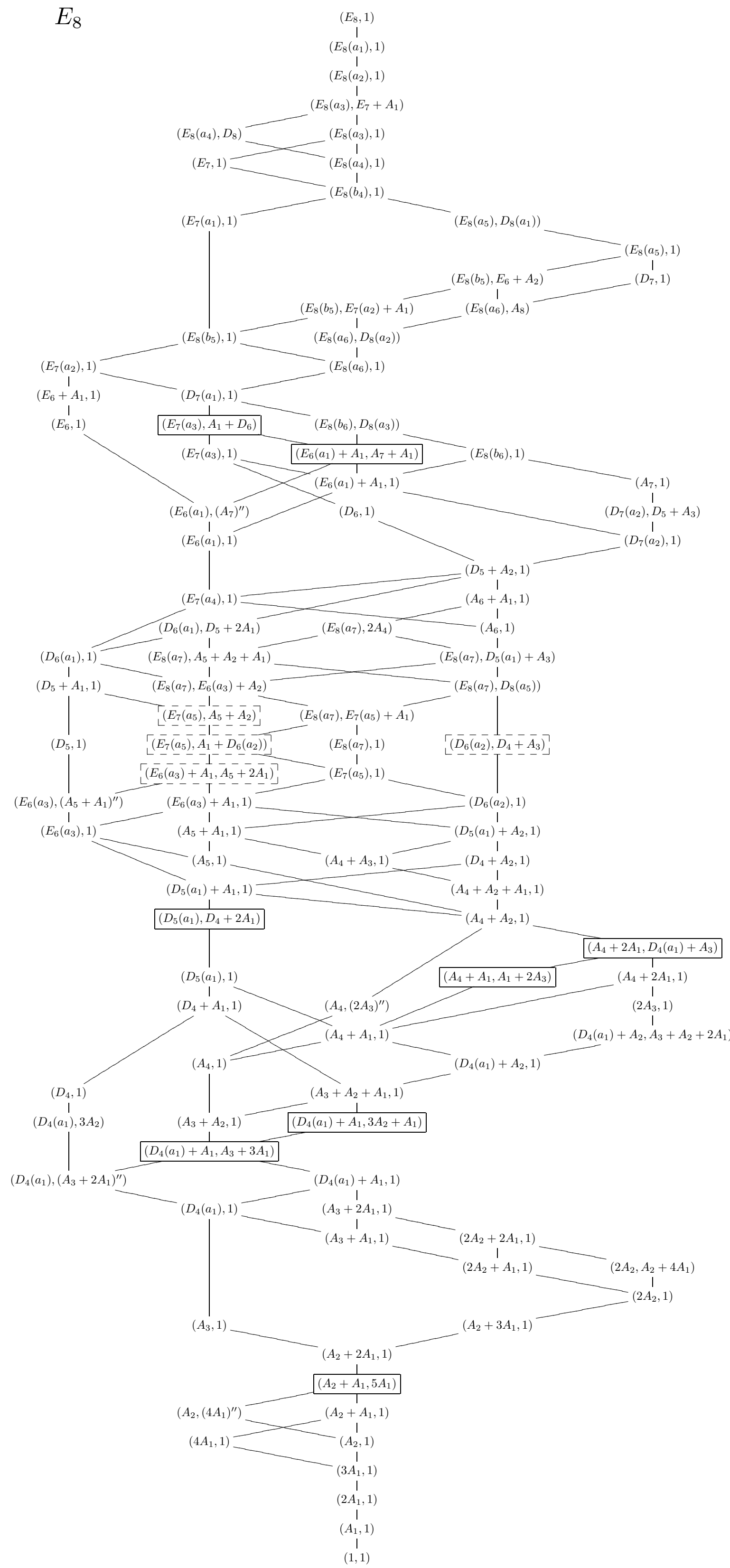
In type E_8, the situation for $(D_6(a_2), D_4 + A_3)$ is just like that for the ambiguous conjugacy classes in F_4; we have recorded both possibilities for $\bar{d}(D_6(a_2), D_4 + A_3)$ above. The situation with the other three ambiguous classes is more complicated, because they occur consecutively in the partial order. To begin with, $d_{\mathrm{S}}(E_7(a_5), A_5 + A_2) = E_6(a_3) + A_1$, and $\bar{A}(E_6(a_3) + A_1) \simeq S_2$, so there are two possibilities for $\bar{d}(E_7(a_5), A_5 + A_2)$. Now, we claim that $\bar{d}(E_6(a_3) + A_1, A_5 + 2A_1) = (E_7(a_5), A_5 + A_2)$ if and only if $\bar{d}(E_7(a_5), A_5 + A_2) = (E_6(a_3) + A_1, A_5 + 2A_1)$: otherwise, axiom (2) will be violated on one of these two classes. In the case that $\bar{d}(E_6(a_3)+A_1, A_5+2A_1) \neq (E_7(a_5), A_5+A_2)$, we have that $(E_6(a_3)+A_1, A_5+2A_1)$ is not special, as there is no other class besides $(E_7(a_5), A_5+A_2)$ that could map to it. Therefore, Corollary 2.5 tells us that $\bar{d}(E_6(a_3)+A_1, A_5+2A_1) = \bar{d}(E_7(a_5), A_1+D_6(a_2))$ in this case. Only $\bar{d}(E_7(a_5), A_1+D_6(a_2))$ is left to consider. This time, $d_{\mathrm{S}}(E_7(a_5), A_1 + D_6(a_2)) = E_7(a_5)$, and $\bar{A}(E_7(a_5)) \simeq S_3$, but the possibility that $\bar{d}(E_7(a_5), A_1 + D_6(a_2)) = (E_7(a_5), A_5 + A_2)$ is immediately ruled out because it would lead to a violation of axiom (2). The two remaining possibilities are given above.

Thus far, we have given two possibilities each for $\bar{d}(E_7(a_5), A_5 + A_2)$, $\bar{d}(E_7(a_5), A_1 + D_6(a_2))$, and $\bar{d}(D_6(a_2), D_4+A_3)$, and we have seen that $\bar{d}(E_6(a_3)+A_1, A_5+2A_1)$ is determined by the above choices. It appears that we have eight possibilities for $\bar{d}$, but when we try to verify that each of these is actually an extended




E_6
$(E_6,1)$
$(E_6(a_1),1)$
$(D_5,1)$
$(E_6(a_3),A_5+A_1)$
$(E_6(a_3),1)$
$(A_5,1)$
$(D_5(a_1),1)$
$(A_4+A_1,1)$
$(D_4,1)$
$(A_4,1)$
$(D_4(a_1),3A_2)$
$(D_4(a_1),A_3+2A_1)$
$(D_4(a_1),1)$
$(A_3+A_1,1)$
$(A_3,1)$
$(2A_2+A_1,1)$
$(A_2+2A_1,1)$
$(2A_2,1)$
$(A_2,4A_1)$
$(A_2+A_1,1)$
$(A_2,1)$
$(3A_1,1)$
$(2A_1,1)$
$(A_1,1)$
$(1,1)$


E_7
$(E_7,1)$
$(E_7(a_1),1)$
$(E_7(a_2),1)$
$(E_6,1)$
$(E_7(a_3),A_1+D_6)$
$(E_6(a_1),A_7)$
$(E_7(a_3),1)$
$(E_6(a_1),1)$
$(D_6,1)$
$(E_7(a_4),1)$
$(A_6,1)$
$(D_5+A_1,1)$
$(D_6(a_1),1)$
$(E_7(a_5),A_5+A_2)$
$(D_5,1)$
$(E_7(a_5),A_1+D_6(a_2))$
$(E_6(a_3),(A_5+A_1)')$
$(E_7(a_5),1)$
$(E_6(a_3),1)$
$(D_6(a_2),1)$
$(A_5+A_1,1)$
$((A_5)',1)$
$(D_5(a_1)+A_1,1)$
$((A_5)'',1)$
$(D_5(a_1),D_4+2A_1)$
$(A_4+A_2,1)$
$(D_5(a_1),1)$
(A_4+A_1,A_1+2A_3)
$(D_4+A_1,1)$
$(A_4+A_1,1)$
$(A_4,2A_3)$
$(A_4,1)$
$(D_4,1)$
$(A_3+A_2+A_1,1)$
$(D_4(a_1),3A_2)$
$(D_4(a_1)+A_1,A_3+3A_1)$
$(A_3+A_2,1)$
$(D_4(a_1),(A_3+2A_1)')$
$(D_4(a_1)+A_1,1)$
$(D_4(a_1),1)$
$(A_3+2A_1,1)$
$((A_3+A_1)',1)$
$(2A_2+A_1,1)$
$((A_3+A_1)'',1)$
$(A_2+3A_1,1)$
$(2A_2,1)$
$(A_3,1)$
$(A_2+2A_1,1)$
$(A_2+A_1,5A_1)$
$(A_2+A_1,1)$
$(A_2,(4A_1)')$
$(4A_1,1)$
$(A_2,1)$
$((3A_1)'',1)$
$((3A_1)',1)$
$(2A_1,1)$
$(A_1,1)$
$(1,1)$


E_8
$(E_8,1)$
$(E_8(a_1),1)$
$(E_8(a_2),1)$
$(E_8(a_3),E_7+A_1)$
$(E_8(a_4),D_8)$
$(E_8(a_3),1)$
$(E_7,1)$
$(E_8(a_4),1)$
$(E_8(b_4),1)$
$(E_7(a_1),1)$
$(E_8(a_5),D_8(a_1))$
$(E_8(a_5),1)$
$(E_8(b_5),E_6+A_2)$
$(D_7,1)$
$(E_8(b_5),E_7(a_2)+A_1)$
$(E_8(a_6),A_8)$
$(E_8(b_5),1)$
$(E_8(a_6),D_8(a_2))$
$(E_7(a_2),1)$
$(E_8(a_6),1)$
$(E_6+A_1,1)$
$(D_7(a_1),1)$
$(E_6,1)$
$(E_7(a_3),A_1+D_6)$
$(E_8(b_6),D_8(a_3))$
$(E_7(a_3),1)$
$(E_6(a_1)+A_1,A_7+A_1)$
$(E_8(b_6),1)$
$(E_6(a_1)+A_1,1)$
$(A_7,1)$
$(E_6(a_1),(A_7)'')$
$(D_6,1)$
$(D_7(a_2),D_5+A_3)$
$(E_6(a_1),1)$
$(D_7(a_2),1)$
$(D_5+A_2,1)$
$(E_7(a_4),1)$
$(A_6+A_1,1)$
$(D_6(a_1),D_5+2A_1)$
$(E_8(a_7),2A_4)$
$(A_6,1)$
$(D_6(a_1),1)$
$(E_8(a_7),A_5+A_2+A_1)$
$(E_8(a_7),D_5(a_1)+A_3)$
$(D_5+A_1,1)$
$(E_8(a_7),E_6(a_3)+A_2)$
$(E_8(a_7),D_8(a_5))$
$(E_7(a_5),A_5+A_2)$
$(E_8(a_7),E_7(a_5)+A_1)$
$(D_5,1)$
$(E_7(a_5),A_1+D_6(a_2))$
$(E_8(a_7),1)$
$(D_6(a_2),D_4+A_3)$
$(E_6(a_3)+A_1,A_5+2A_1)$
$(E_7(a_5),1)$
$(E_6(a_3),(A_5+A_1)'')$
$(E_6(a_3)+A_1,1)$
$(D_6(a_2),1)$
$(E_6(a_3),1)$
$(A_5+A_1,1)$
$(D_5(a_1)+A_2,1)$
$(A_5,1)$
$(A_4+A_3,1)$
$(D_4+A_2,1)$
$(D_5(a_1)+A_1,1)$
$(A_4+A_2+A_1,1)$
$(D_5(a_1),D_4+2A_1)$
$(A_4+A_2,1)$
$(A_4+2A_1,D_4(a_1)+A_3)$
$(D_5(a_1),1)$
(A_4+A_1,A_1+2A_3)
$(A_4+2A_1,1)$
$(D_4+A_1,1)$
$(A_4,(2A_3)'')$
$(2A_3,1)$
$(A_4+A_1,1)$
$(D_4(a_1)+A_2,A_3+A_2+2A_1)$
$(A_4,1)$
$(D_4(a_1)+A_2,1)$
$(D_4,1)$
$(A_3+A_2+A_1,1)$
$(D_4(a_1),3A_2)$
$(A_3+A_2,1)$
$(D_4(a_1)+A_1,3A_2+A_1)$
$(D_4(a_1)+A_1,A_3+3A_1)$
$(D_4(a_1),(A_3+2A_1)'')$
$(D_4(a_1)+A_1,1)$
$(D_4(a_1),1)$
$(A_3+2A_1,1)$
$(A_3+A_1,1)$
$(2A_2+2A_1,1)$
$(2A_2+A_1,1)$
$(2A_2,A_2+4A_1)$
$(2A_2,1)$
$(A_3,1)$
$(A_2+3A_1,1)$
$(A_2+2A_1,1)$
$(A_2+A_1,5A_1)$
$(A_2,(4A_1)'')$
$(A_2+A_1,1)$
$(4A_1,1)$
$(A_2,1)$
$(3A_1,1)$
$(2A_1,1)$
$(A_1,1)$
$(1,1)$

duality map, we encounter one more restriction. If we have $\bar{d}(E_7(a_5), A_5 + A_2) = (E_6(a_3) + A_1, A_5 + 2A_1)$, then putting $\bar{d}(E_7(a_5), A_1 + D_6(a_2)) = (E_7(a_5), 1)$ results in a violation of axiom (1). That cuts down the number of possibilities to six, each of which is, in fact, an extended duality map. $\square$

Above, we have drawn out the full Hasse diagram of the partial-order structure on $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$ in types B and C up to rank 4, in type D up to rank 5, and in all the exceptional groups. In these diagrams, most pairs $(\mathcal{O}, C)$ are special. Ones that are not special are indicated by a solid box $\boxed{}$. In types F_4 and E_8, where we have multiple choices for $\bar{d}$, a dashed box surrounds pairs on which $\bar{d}$ is not uniquely determined. Note that every such pair turns out to be special for some choices of $\bar{d}$ and nonspecial for other choices.

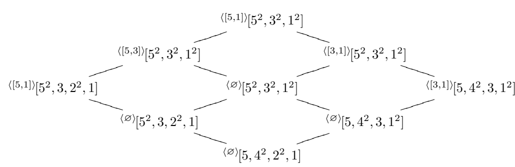
In type D and the exceptional groups, the duality map $\bar{d}$ itself can be visualized as follows: if the nonspecial pairs are deleted from the diagram, the remaining partial-order diagram has a horizontal axis of symmetry. The duality map on special pairs is given by reflection across this axis; then, Proposition 2.4 tells us how to compute $\bar{d}$ on nonspecial pairs. For types B and C, we have drawn the Hasse diagram of $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}(B_n)$ directly above that of $\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}(C_n)$. This combined diagram has a horizontal axis of symmetry if nonspecial pairs are deleted, and $\bar{d}$ is given by reflecting across that.

7. Further comments

It has been observed that the duality map is uniquely determined in some of the exceptional groups; the computed examples show that the same holds in many classical groups of small rank. What can be said towards a general uniqueness statement for this map?

Spaltenstein, in his original axiomatic presentation of d_{LS} [17], proved that that map was uniquely determined in the classical groups, but found that there were multiple possibilities in every exceptional group but E_7. He notes, however, that the duality map is uniquely determined if one adds the further requirement that the set of special orbits be as small as possible. He does not provide much in the way of motivation for this particular requirement, but it turns out to be the "right" one, in that the set of special orbits thus obtained agrees with those obtained by other approaches to specialness, for instance via the Springer correspondence.

However, our $\bar{d}$ is not the unique map satisfying the axioms of Section 1 even in the classical groups. A portion of the partial-order diagram in D_9 is shown here. This portion is from the middle of the Hasse diagram; the duality map is given by reflecting across the horizontal axis of symmetry,



At first glance, the axioms seem to determine the value of the duality map on all the conjugacy classes shown here, with the exception of ${}^{\langle[5,1]\rangle}[5^2, 3, 2^2, 1]$ and ${}^{\langle[3,1]\rangle}[5, 4^2, 3, 1^2]$. However, if we drew in slightly more of the picture, we would see that there is not a unique smallest special marked partition larger than ${}^{\langle[5,1]\rangle}[5^2, 3, 2^2, 1]$, so by Proposition 2.4, the duality map is determined on that class as well: that class must be special and self-dual.

But ${}^{\langle[3,1]\rangle}[5, 4^2, 3, 1^2]$ has the property that any class larger than it is also larger than ${}^{\langle[3,1]\rangle}[5^2, 3^2, 1^2]$. It would be consistent with the axioms to have $\bar{d}({}^{\langle[3,1]\rangle}[5, 4^2, 3, 1^2]) = {}^{\langle\varnothing\rangle}[5, 4^2, 3, 1^2]$, but in fact we have $\bar{d}({}^{\langle[3,1]\rangle}[5, 4^2, 3, 1^2]) = {}^{\langle[3,1]\rangle}[5, 4^2, 3, 1^2]$. Our map $\bar{d}$ is therefore not the only map satisfying the axioms; moreover, it does not have the property that its special set is of minimal size.

The best solution to the lack of a uniqueness statement is probably to give an entirely different approach to the duality map altogether. It would be nice to have a more intrinsic, representation-theoretic description of it, perhaps in terms of the Springer correspondence or the generalized Bala-Carter classification of conjugacy classes in $A(\mathcal{O})$-groups. We now consider some background that might suggest where such a representation-theoretic description might come from.

Assume that G is simply connected, and let $\mathcal{N}_{\mathrm{o,r}}$ be the set of pairs $\{(\mathcal{O}, \rho)\}$, where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$, and ρ is an irreducible representation of the isotropy group of $\mathcal{O}$ in G. Let $\mathcal{N}^0_{\mathrm{o,r}}$ be similarly defined, except that we take ρ to be a representation of the fundamental group $\pi_1(\mathcal{O})$ instead. This fundamental group is just the component group of the isotropy group. We can regard $\mathcal{N}^0_{\mathrm{o,r}}$ as a subset of $\mathcal{N}_{\mathrm{o,r}}$, since any irreducible representation of $\pi_1(\mathcal{O})$ can be pulled back to give an irreducible representation of the isotropy group.

Lusztig conjectured the existence of a natural bijection between $\mathcal{N}_{\mathrm{o,r}}$ and the set Λ_+ of dominant weights of G, that should arise by studying the equivariant K-theory of the nilpotent cone [12]. This idea has been investigated by Bezrukavnikov [5], [6], Ostrik [14], and the author [1]. Additionally, Chmutova and Ostrik have computed tables of the weights that this bijection should assign to pairs $(\mathcal{O},\rho)$ when ρ is trivial [8]. Now, nilpotent orbits in ${}^L\mathfrak{g}$ are labelled by their weighted Dynkin diagrams, which may be regarded as weights for G. (The weighted Dynkin diagram of the orbit is the semisimple element of the Jacobson-Morozov $\mathfrak{sl}(2)$-triple for the orbit.) It has been observed that, given an orbit $\mathcal{O} \in {}^L\mathcal{N}_{\mathrm{o}}$, this bijection often sends its weighted Dynkin diagram to some pair $(\mathcal{O}',\rho)$, where $\mathcal{O}' = d_{\mathrm{BV}}(\mathcal{O})$ and, moreover, ρ is a representation that descends to the group $\pi_1(\mathcal{O})$. This is mentioned in [8]; a more thorough discussion can be found in Section 3 of [2]. Versions of the following conjecture appear in those sources.

Conjecture 7.1. *There is a map $\delta : {}^L\mathcal{N}_{\mathrm{o}} \to \mathcal{N}^0_{\mathrm{o,r}}$, such that $pr_1 \circ \delta = d_{\mathrm{BV}}$, and such that the aforementioned bijection attaches the Dynkin diagram of $\mathcal{O} \in {}^L\mathcal{N}_{\mathrm{o}}$ to the pair $\delta(\mathcal{O})$.*

This statement has been proved in type A and verified by explicit computation in a number of other groups of small rank, but it is not known in general. In [2], it is observed that one cannot readily guess the representation ρ of $\pi_1(d_{\mathrm{BV}}(\mathcal{O}))$ to which the weighted Dynkin diagram of $\mathcal{O}$ should be assigned.

The present work actually began as an attempt by the author to impose a partial order on the set $\mathcal{N}^0_{\mathrm{o,r}}$, with the hope that that partial order might shed some light on what the conjectural map δ should look like. The discussion in [2] includes a construction of a particular cover $\tilde{\mathcal{O}}$ of a given orbit $\mathcal{O}$ associated to a pair $(\mathcal{O},C) \in \mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}}$. One can then push forward the structure sheaf of $\tilde{\mathcal{O}}$ to get a G-equivariant coherent sheaf on $\mathcal{O}$; in fact, this sheaf is a sum of irreducible local systems on $\mathcal{O}$. Local systems correspond to representations of $\pi_1(\mathcal{O})$, so to each pair $(\mathcal{O},C)$, there is an associated (not necessarily irreducible) representation of $\pi_1(\mathcal{O})$.

This set-up allows us to refine the previous conjecture.

Conjecture 7.2. *There is a natural partial order on $\mathcal{N}^0_{\mathrm{o,r}}$, with respect to which there is a unique "largest" irreducible representation ρ that occurs as a constituent of the representation of $\pi_1(\mathcal{O})$ associated to $(\mathcal{O},C)$. If $(\mathcal{O},C)$ is Sommers' canonical inverse for ${}^L\mathcal{O} \in {}^L\mathcal{N}_{\mathrm{o}}$, then the above map $\delta : {}^L\mathcal{N}_{\mathrm{o}} \to \mathcal{N}^0_{\mathrm{o,r}}$ is an order-reversing map given by $\mathcal{O}' \mapsto (\mathcal{O},\rho)$.*

The putative partial order on $\mathcal{N}^0_{\mathrm{o,r}}$ may not be very easy to characterize. For instance, let $\mathcal{O}$ be the minimal orbit in type C_n; we have $\pi_1(\mathcal{O}) \simeq \mathbb{Z}/2\mathbb{Z}$. Sommers has observed that the larger of the two representations of $\pi_1(\mathcal{O})$ ought to be the trivial one when n is odd, but the nontrivial one when n is even ([2], Remark 3.2). In [3], Aubert defines a partial order on the set of "u-symbols," introduced by Lusztig [11] as a way to parametrize $\mathcal{N}^0_{\mathrm{o,r}}$ in the classical groups. It is not known whether this might be the partial order sought above.

Finally, assuming the issues raised above have been understood, one can construct a (possibly order-reversing?) map ${}^L\mathcal{N}_{\mathrm{o},\bar{\mathrm{c}}} \to \mathcal{N}^0_{\mathrm{o,r}}$, because the way in which a representation of $\pi_1(\mathcal{O})$ is produced does not depend on $(\mathcal{O},C)$ being the canonical inverse of some orbit $\mathcal{O}'$. What might be the representation-theoretic meaning of such a map? Can it also be interpreted as saying something about Lusztig's bijection?

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