PARITY SHEAVES ON THE AFFINE GRASSMANNIAN AND THE MIRKOVIĆ–VILONEN CONJECTURE

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Abstract. We prove the Mirković–Vilonen conjecture: the integral local intersection cohomology groups of spherical Schubert varieties on the affine Grassmannian have no p-torsion, as long as p is outside a certain small and explicitly given set of prime numbers. (Juteau has exhibited counterexamples when p is a bad prime.) The main idea is to convert this topological question into an algebraic question about perverse-coherent sheaves on the dual nilpotent cone using the Juteau–Mautner–Williamson theory of parity sheaves.

1. Introduction

1.1. **Overview.** Let G be a connected complex reductive group, and let $\mathcal{G}r$ denote its affine Grassmannian. This space has the remarkable property that its topology encodes the representation theory of the split Langlands dual group $G^\vee$ over any field $\Bbbk$ (or even over a commutative ring). To be more precise, the *geometric Satake equivalence*, in the form due to Mirković–Vilonen [MV2] (see also [L, G2]), asserts that there is an equivalence of tensor categories

$$\mathcal{S} : \mathrm{Rep}(G^\vee) \xrightarrow{\sim} \mathrm{Perv}_{G_{\mathbf{O}}}(\mathcal{G}r, \Bbbk) \tag{1.1}$$

where $\mathrm{Perv}_{G_{\mathbf{O}}}(\mathcal{G}r, \Bbbk)$ is the category of spherical perverse $\Bbbk$-sheaves on $\mathcal{G}r$. (A full explanation of the notation is given in Section 1.2 below.) This result raises the possibility of comparing representation theory over different fields via the universal coefficient theorem of topology.

For instance, let λ be a dominant coweight for G, and let $\mathcal{I}_!(\lambda, \Bbbk)$ denote the "standard" perverse sheaf on the corresponding stratum of $\mathcal{G}r$. This perverse sheaf serves as a topological realization of a Weyl module for $G^\vee$. When $\Bbbk = \mathbb{C}$, it is simple, and its stalks are described by Kazhdan–Lusztig theory.

With a view to applications in modular representation theory, Mirković and Vilonen conjectured in the late 1990s [MV1] that that the stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ are torsion-free. This implies that the $\Bbbk$-stalks are "independent" of $\Bbbk$. Their conjecture was slightly too optimistic: counterexamples due to Juteau [Ju] reveal the presence of torsion, but only at bad primes. Juteau proposed a modified conjecture, asserting that there is no p-torsion as long as p is a good prime for G. In this paper, we prove the following result, confirming this conjecture in nearly all cases.

Theorem 1.1. *If p is a JMW prime for G* (see Table 1)*, then the stalks of $\mathcal{I}_!(\lambda, \mathbb{Z})$ have no p-torsion. Furthermore, if $\Bbbk$ is a field whose characteristic is a JMW prime, then the stalks of $\mathcal{I}_!(\lambda, \Bbbk)$ have a parity-vanishing property.*

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Type	Bound	Type	Bound
A_n	any p	E_6, F_4, G_2	$p > 3$
B_n	$p > n$	E_7	$p > 19$
C_n, D_n	$p > 2$	E_8	$p > 31$

TABLE 1. Currently known bounds for JMW primes

An outline of the proof will be explained below, after some preliminaries.

1.2. **The constructible side.** Recall that $\mathcal{G}r = G_{\mathbf{K}}/G_{\mathbf{O}}$, where $\mathbf{K} = \mathbb{C}((t))$ and $\mathbf{O} = \mathbb{C}[[t]]$. For the remainder of Section 1, $\Bbbk$ will denote an algebraically closed field. Let $\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r, \Bbbk)$ denote the bounded derived category of complexes of $\Bbbk$-sheaves on $\mathcal{G}r$ that are constructible with respect to the $G_{\mathbf{O}}$-orbits, and let $\mathrm{Perv}_{G_{\mathbf{O}}}(\mathcal{G}r, \Bbbk) \subset \mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r, \Bbbk)$ be the subcategory of perverse sheaves. Those $G_{\mathbf{O}}$-orbits are naturally in bijection with the set $\mathbf{X}^+$ of dominant coweights for G. For $\lambda \in \mathbf{X}^+$, let $i_\lambda : \mathcal{G}r_\lambda \hookrightarrow \mathcal{G}r$ be the inclusion map of the corresponding orbit.

For $\lambda \in \mathbf{X}^+$, the irreducible (resp. Weyl, dual Weyl, indecomposable tilting) $G^\vee$-module of highest weight λ is denoted by $\mathrm{L}(\lambda)$ (resp. $\mathrm{M}(\lambda)$, $\mathrm{N}(\lambda)$, $\mathrm{T}(\lambda)$). The perverse sheaves corresponding to these objects under $\mathcal{S}$ are denoted by $\mathrm{IC}(\lambda)$, (resp. $\mathcal{I}_!(\lambda)$, $\mathcal{I}_*(\lambda)$, $\mathcal{T}(\lambda)$). Of course, $\mathrm{IC}(\lambda)$ is a simple perverse sheaf. We saw $\mathcal{I}_!(\lambda)$ earlier; $\mathcal{I}_*(\lambda)$ is its Verdier dual, a costandard perverse sheaf.

What about the $\mathcal{T}(\lambda)$? It is a deep insight of Juteau–Mautner–Williamson that these perverse sheaves should be characterized by a topological property: specifically, they ought to be *parity sheaves* in the sense of [JMW].

Definition 1.2. A prime number p is said to be a *JMW prime* for G if it is good for G and, whenever $\Bbbk$ has characteristic p, each $\mathcal{T}(\lambda)$ is a parity sheaf on $\mathcal{G}r$.

Juteau, Mautner, and Williamson proved[1] that the primes in Table 1 are JMW primes, and they conjectured that every good prime is a JMW prime. Our proof of Theorem 1.1 treats this notion as an assumption: if additional primes are shown to be JMW in the future, then Theorem 1.1 will apply to those as well.

1.3. **The coherent side.** The main idea of the proof of Theorem 1.1 is to translate the problem into an algebraic question about coherent sheaves on the nilpotent cone $\mathcal{N}$ for $G^\vee$. The motivation comes from an old result of Ginzburg [G2, Proposition 1.10.4]: when $\Bbbk = \mathbb{C}$, he showed that for all $V_1, V_2 \in \mathrm{Rep}(G^\vee)$, there is an isomorphism of graded vector spaces

$$\mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r,\Bbbk)}(\mathcal{S}(V_1), \mathcal{S}(V_2)) \cong \mathrm{Hom}^\bullet_{\mathrm{Coh}^{G^\vee \times \mathbb{G}_{\mathrm{m}}}(\mathcal{N})}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}). \tag{1.2}$$

For details on the category $\mathrm{Coh}^{G^\vee \times \mathbb{G}_{\mathrm{m}}}(\mathcal{N})$, see Section 2.4.

To imitate this in positive characteristic, we need control over the algebraic geometry of $\mathcal{N}$. This is achieved by the following condition.

Definition 1.3. A prime p is said to be *rather good* for $G^\vee$ if it is both good for $G^\vee$ and coprime to the order of the fundamental group of the derived group of $G^\vee$.

[1] See [JMWv1, Theorem 5.1]. This result was removed from the later revision [JMW] for inclusion in a separate forthcoming paper. Note that for $G^\vee$ of type B_n or D_n, [JMWv1, Theorem 5.1] gives the bounds $p > n - 1$ or $p > n - 2$, respectively. According to [JMWv1, Remark 5.3], however, these can be improved to $p > 2$.

When $G^\vee$ is semisimple, "rather good" coincides with "good" if $G^\vee$ is simply connected, and with "very good" if $G^\vee$ is of adjoint type. When $\operatorname{char} \Bbbk$ is rather good for $G^\vee$, it is feasible to adapt Ginzburg's argument, provided that $\mathcal{S}(V_1)$ and $\mathcal{S}(V_2)$ are parity.[2]

To push this result further, we need the following observation: coherent sheaves of the form $V \otimes \mathcal{O}_{\mathcal{N}}$ also lie in the category of *perverse-coherent sheaves*, denoted $\mathrm{PCoh}^{G^\vee \times \mathbb{G}_\mathrm{m}}(\mathcal{N})$, or simply $\mathrm{PCoh}(\mathcal{N})$. This category, which is the heart of a certain t-structure on $\mathrm{D}^\mathrm{b}\mathrm{Coh}^{G^\vee \times \mathbb{G}_\mathrm{m}}(\mathcal{N})$, looks very different from $\mathrm{Coh}^{G^\vee \times \mathbb{G}_\mathrm{m}}(\mathcal{N})$. For instance, every object of $\mathrm{PCoh}(\mathcal{N})$ has finite length. We will not use the details of its definition in this paper; we just require a structural property discussed in Section 2.4.

Interpreting the right-hand side of (1.2) as a Hom-group in $\mathrm{PCoh}(\mathcal{N})$ leads to new avenues for generalizing that result. For $\mu \in \mathbf{X}^+$, let $\mathrm{PCoh}(\mathcal{N})_{\leq \mu} \subset \mathrm{PCoh}(\mathcal{N})$ be the Serre subcategory generated by $\mathrm{N}(\nu) \otimes \mathcal{O}_{\mathcal{N}}\langle n \rangle$ with $\nu \leq \mu$. (Here, $\langle n \rangle$ indicates a twist of the $\mathbb{G}_\mathrm{m}$-action.) In Section 5, we prove the following result, which seems to be new even for $\Bbbk = \mathbb{C}$.

Theorem 1.4. *Suppose that $V_1 \in \mathrm{Rep}(G^\vee)$ has a Weyl filtration, and that $V_2 \in \mathrm{Rep}(G^\vee)$ has a good filtration. Let $j : \mathcal{G}r \setminus \overline{\mathcal{G}r_\lambda} \to \mathcal{G}r$ be the inclusion map. If p is rather good for $G^\vee$ and JMW for G, then there is a natural isomorphism*

$$\mathrm{Hom}^\bullet(j^*\mathcal{S}(V_1), j^*\mathcal{S}(V_2)) \cong \mathrm{Hom}^\bullet(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})),$$

where $\Pi : \mathrm{PCoh}(\mathcal{N}) \to \mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_{\leq\lambda}$ is the Serre quotient functor.

Intuitively, this theorem gives us an algebraic counterpart in $\mathrm{D}^\mathrm{b}\mathrm{Coh}^{G^\vee \times \mathbb{G}_\mathrm{m}}(\mathcal{N})$ of the geometric notion of "restricting to an open subset" in $\mathcal{G}r$. Once we have that, it is not difficult to translate the problem of studying stalks of $\mathcal{I}_!(\lambda)$ into an algebraic question about certain objects in $\mathrm{PCoh}(\mathcal{N})$ and its quotients. The latter question turns out to be quite easy (see Lemma 2.12).

1.4. **Outline of the paper.** In Section 2, we recall the necessary background on properly stratified categories and on $\mathrm{PCoh}(\mathcal{N})$, largely following the work of Minn-Thu-Aye. Section 3 reviews the theory of parity sheaves. In Section 4, which can be read independently of the rest of the paper, we study the cohomology of parity sheaves on flag varieties of Kac–Moody groups, generalizing earlier results of Soergel and Ginzburg. That result is a step on the way to Theorem 1.4, which is proved in Section 5. Finally, the main result, Theorem 1.1, is proved in Section 6.

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2. Properly stratified categories

2.1. **Definition and background.** Let $\Bbbk$ be a field, and let $\mathscr{C}$ be a $\Bbbk$-linear abelian category in which every object has finite length. Assume that $\mathscr{C}$ is equipped with an

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[2]When $\Bbbk = \mathbb{C}$, the identification of Ext-groups in (1.2) is a precursor of a derived equivalence relating the Satake category to coherent sheaves on the nilpotent cone [ABG]. We warn the reader that the obvious modification of that equivalence for general $\Bbbk$ is *not* expected to be true by experts. More care is needed to formulate the correct analogue in the modular setting.

automorphism $\langle 1\rangle : \mathscr{C} \to \mathscr{C}$, which we will refer to as the *Tate twist*. For $X, Y \in \mathscr{C}$, let $\underline{\mathrm{Hom}}(X,Y)$ be the graded vector space given by

$$\underline{\mathrm{Hom}}(X,Y)_n = \mathrm{Hom}(X, Y\langle -n\rangle).$$

The Tate twist induces an action of $\mathbb{Z}$ on the set $\mathrm{Irr}(\mathscr{C})$ of isomorphism classes of simple objects in $\mathscr{C}$. Assume that this action is free, and let $\Omega = \mathrm{Irr}(\mathscr{C})/\mathbb{Z}$. For each $\gamma \in \Omega$, choose a representative simple object $L_\gamma \in \mathscr{C}$ whose isomorphism class lies in the $\mathbb{Z}$-orbit $\gamma \subset \mathrm{Irr}(\mathscr{C})$. Thus, every simple object in $\mathscr{C}$ is isomorphic to some $L_\gamma\langle n\rangle$ with $\gamma \in \Omega$ and $n \in \mathbb{Z}$.

Assume that Ω is equipped with a partial order $\leq$, and that for any $\gamma \in \Omega$, the set $\{\xi \in \Omega \mid \xi \leq \gamma\}$ is finite. For any order ideal $\Gamma \subset \Omega$, let $\mathscr{C}_\Gamma \subset \mathscr{C}$ be the Serre subcategory generated by the simple objects $\{L_\gamma\langle n\rangle \mid \gamma \in \Gamma,\ n \in \mathbb{Z}\}$. (Recall that an *order ideal* is a subset $\Gamma \subset \Omega$ such that if $\gamma \in \Gamma$ and $\xi \leq \gamma$, then $\xi \in \Gamma$.) As a special case, we write

$$\mathscr{C}_{\leq\gamma} = \mathscr{C}_{\{\xi\in\Omega \mid \xi\leq\gamma\}}. \tag{2.1}$$

The category $\mathscr{C}_{<\gamma}$ is defined similarly.

Definition 2.1. Suppose $\mathscr{C}$, Ω, and $\leq$ are as above. We say that $\mathscr{C}$ is a *graded properly stratified category* if for each $\gamma \in \Omega$, the following conditions hold:

(1) We have $\mathrm{End}(L_\gamma) \cong \Bbbk$.
(2) There is an object $\bar\Delta_\gamma$ and a surjective morphism $\phi_\gamma : \bar\Delta_\gamma \to L_\gamma$ such that

$$\ker(\phi_\gamma) \in \mathscr{C}_{<\gamma} \qquad \text{and} \qquad \underline{\mathrm{Hom}}(\bar\Delta_\gamma, L_\xi) = \underline{\mathrm{Ext}}^1(\bar\Delta_\gamma, L_\xi) = 0 \text{ if } \xi \not\geq \gamma.$$

(3) There is an object $\bar\nabla_\gamma$ and an injective morphism $\psi_\gamma : L_\gamma \to \bar\nabla_\gamma$ such that

$$\mathrm{cok}(\psi_\gamma) \in \mathscr{C}_{<\gamma} \qquad \text{and} \qquad \underline{\mathrm{Hom}}(L_\xi, \bar\nabla_\gamma) = \underline{\mathrm{Ext}}^1(L_\xi, \bar\nabla_\gamma) = 0 \text{ if } \xi \not\geq \gamma.$$

(4) In $\mathscr{C}_{\leq\gamma}$, L_γ admits a projective cover $\Delta_\gamma \to L_\gamma$. Moreover, Δ_γ admits a filtration whose subquotients are of the form $\bar\Delta_\gamma\langle n\rangle$ for various $n \in \mathbb{Z}$.
(5) In $\mathscr{C}_{\leq\gamma}$, L_γ admits an injective envelope $L_\gamma \to \nabla_\gamma$. Moreover, ∇_γ admits a filtration whose subquotients are of the form $\bar\nabla_\gamma\langle n\rangle$ for various $n \in \mathbb{Z}$.
(6) We have $\underline{\mathrm{Ext}}^2(\Delta_\gamma, \bar\nabla_\xi) = \underline{\mathrm{Ext}}^2(\bar\Delta_\gamma, \nabla_\xi) = 0$ for all $\gamma, \xi \in \Omega$.

An object in $\mathscr{C}$ is said to be *standard* (resp. *costandard*, *proper standard*, *proper costandard*) if it is isomorphic to some $\Delta_\gamma\langle n\rangle$ (resp. $\nabla_\gamma\langle n\rangle$, $\bar\Delta_\gamma\langle n\rangle$, $\bar\nabla_\gamma\langle n\rangle$).

More generally, a *standard* (resp. *costandard*, *proper standard*, *proper costandard*) filtration of an object of $\mathscr{C}$ is a filtration whose subquotients are all standard (resp. costandard, proper standard, proper costandard) objects.

Routine arguments (see [B, Lemma 1]) show that when objects $\bar\Delta_\gamma$, $\bar\nabla_\gamma$, Δ_γ, ∇_γ with the above properties exist, they are unique up to isomorphism. It may happen that $\bar\Delta_\gamma \cong \Delta_\gamma$ and $\bar\nabla_\gamma \cong \nabla_\gamma$; in that case, $\mathscr{C}$ is usually called a *highest weight* or *quasi-hereditary category.* The class of objects in $\mathscr{C}$ admitting a standard (resp. costandard, proper standard, proper costandard) filtration is denoted

$$\mathscr{F}(\Delta), \qquad \text{resp.} \qquad \mathscr{F}(\nabla), \qquad \mathscr{F}(\bar\Delta), \qquad \mathscr{F}(\bar\nabla).$$

The relationship between the notions above and the notion of a *properly stratified algebra* [D, FM] is explained in [Mi]. In particular, results in [Mi] explain how to transfer results from the literature on properly stratified algebras to our setting. For instance, the following result is a restatement of [D, Definition 4 and Theorem 5].

Proposition 2.2. *Let $\Gamma \subset \Omega$ be a finite order ideal. Then the Serre quotient $\mathscr{C}/\mathscr{C}_\Gamma$ is again a graded properly stratified category, and $\mathrm{Irr}(\mathscr{C}/\mathscr{C}_\Gamma)/\mathbb{Z}$ is naturally identified with $\Omega \setminus \Gamma$. Indeed, we have a recollement diagram*

$$\begin{tikzcd} \mathrm{D^b}\mathscr{C}_\Gamma \arrow[r, "\imath"] & \mathrm{D^b}\mathscr{C} \arrow[l, bend right, "\imath^{\mathrm{L}}"'] \arrow[l, bend left, "\imath^{\mathrm{R}}"] \arrow[r, "\Pi"] & \mathrm{D^b}(\mathscr{C}/\mathscr{C}_\Gamma) \arrow[l, bend right, "\Pi^{\mathrm{L}}"'] \arrow[l, bend left, "\Pi^{\mathrm{R}}"] \end{tikzcd}$$

Here, the superscripts L and R indicate the left and right adjoints, respectively, of $\imath$ and Π. An important property implied by the preceding proposition is that

$$\underline{\mathrm{Ext}}^k(\Delta_\gamma, \bar\nabla_\xi) = \underline{\mathrm{Ext}}^k(\bar\Delta_\gamma, \nabla_\xi) = 0 \qquad \text{for all } k > 0.$$

Also implicit in Proposition 2.2 (or explicit in its proof) are the next two lemmas, which express the compatibility of the various functors with the properly stratified structure. For analogues in the quasi-hereditary case, see [CPS].

Lemma 2.3. *The functors $\imath$ and Π are t-exact and preserve the property of having a standard (resp. costandard, proper standard, proper costandard) filtration.*

The remaining functors in the recollement diagram are not t-exact in general, but they do send certain classes of objects to the heart of the t-structure.

Lemma 2.4. *The functors $\imath^{\mathrm{L}}$ and Π^{L} preserve the property of having a standard or proper standard filtration. The functors $\imath^{\mathrm{R}}$ and Π^{R} preserve the property of having a costandard or proper costandrd filtration.*

2.2. **Tilting objects.** In contrast with the quasi-hereditary case, there are, in general, two inequivalent notions of "tilting" in a properly stratified category.

Definition 2.5. A *tilting object* is an object in $\mathscr{F}(\Delta) \cap \mathscr{F}(\bar\nabla)$. A *cotilting object* is an object in $\mathscr{F}(\bar\Delta) \cap \mathscr{F}(\nabla)$.

The next proposition gives the classification of indecomposable tilting and cotilting objects. (See [AHLU] for a similar statement for properly stratified algebras.)

Proposition 2.6. *For each $\gamma \in \Omega$, there is an indecomposable tilting object T_γ, unique up to isomorphism, that fits into short exact sequences*

$$0 \to \Delta_\gamma \to T_\gamma \to X \to 0 \qquad \text{and} \qquad 0 \to Y \to T_\gamma \to \bar\nabla_\gamma \to 0$$

with $X \in \mathscr{F}(\Delta)_{<\gamma}$ and $Y \in \mathscr{F}(\bar\nabla)_{\leq\gamma}$. Dually, there is an indecomposable cotilting object T'_γ, unique up to isomorphism, with short exact sequences

$$0 \to \bar\Delta_\gamma \to T'_\gamma \to X' \to 0 \qquad \text{and} \qquad 0 \to Y' \to T'_\gamma \to \nabla_\gamma \to 0$$

with $X' \in \mathscr{F}(\bar\Delta)_{\leq\gamma}$ and $Y' \in \mathscr{F}(\nabla)_{<\gamma}$. Moreover, every indecomposable tilting (resp. cotilting) object is isomorphic to some $T_\gamma\langle n\rangle$ (resp. $T'_\gamma\langle n\rangle$).

Lemma 2.7. *Assume that the tilting and cotilting objects in $\mathscr{C}$ coincide. Then:*

1. *If $\gamma \in \Omega$ is minimal, then $\Delta_\gamma \cong T_\gamma \cong \nabla_\gamma$.*
2. *For any $\gamma \in \Omega$, we have $\underline{\mathrm{Ext}}^1(\nabla_\gamma, \bar\nabla_\gamma) = 0$.*
3. *For any $\gamma \in \Omega$, there is an isomorphism $\underline{\mathrm{Hom}}(\nabla_\gamma, \nabla_\gamma) \xrightarrow{\sim} \underline{\mathrm{Hom}}(\Delta_\gamma, \nabla_\gamma)$.*

Proof. (1) This is immediate from the short exact sequences in Proposition 2.6.

(2) Consider the long exact sequence

$$\cdots \to \underline{\mathrm{Hom}}(Y', \bar{\nabla}_\gamma) \to \underline{\mathrm{Ext}}^1(\nabla_\gamma, \bar{\nabla}_\gamma) \to \underline{\mathrm{Ext}}^1(T_\gamma, \bar{\nabla}_\gamma) \to \cdots$$

The first term vanishes because $Y' \in \mathscr{C}_{<\gamma}$, and the last term vanishes because $T_\gamma \in \mathscr{F}(\Delta)$. The result follows.

(3) It is easy to see that the natural maps $\underline{\mathrm{Hom}}(\nabla_\gamma, \nabla_\gamma) \to \underline{\mathrm{Hom}}(T_\gamma, \nabla_\gamma)$ and $\underline{\mathrm{Hom}}(T_\gamma, \nabla_\gamma) \to \underline{\mathrm{Hom}}(\Delta_\gamma, \nabla_\gamma)$ are both isomorphisms. □

Proposition 2.8 ([Mi]; cf. [BBM, Proposition 1.5])**.** *Assume that the tilting and cotilting objects in $\mathscr{C}$ coincide. Let $\mathcal{T} \subset \mathscr{C}$ be the full subcategory of tilting objects, and consider its homotopy category $\mathrm{K}^{\mathrm{b}}\mathcal{T}$. The obvious functor*

$$\mathrm{K}^{\mathrm{b}}\mathcal{T} \to \mathrm{D}^{\mathrm{b}}\mathscr{C} \tag{2.2}$$

is fully faithful. In case $\mathscr{C}$ is quasi-hereditary, it is an equivalence.

Proposition 2.9. *Assume that the tilting and cotilting objects in $\mathscr{C}$ coincide. The following conditions are equivalent:*

1. *$X \in \mathscr{F}(\Delta)$.*
2. *There is an exact sequence $0 \to X \to T^0 \to T^1 \to \cdots \to T^k \to 0$ where all the T^i are tilting.*

Before proving this, we record one immediate consequence.

Definition 2.10. For $X \in \mathscr{F}(\Delta)$, we define the *tilting dimension* of X, denoted $\mathrm{tdim}\, X$, to be the smallest integer k such that there exists a resolution of X of length k by tilting objects, as in Proposition 2.9.

Corollary 2.11. *If $X \in \mathscr{F}(\Delta)$, there is a short exact sequence*

$$0 \to X \to T \to X' \to 0 \tag{2.3}$$

where T is tilting, $X' \in \mathscr{F}(\Delta)$, and $\mathrm{tdim}\, X' = \mathrm{tdim}\, X - 1$.

Proof of Proposition 2.9. Let $\mathscr{F}(\Delta)'$ be the class of objects X satisfying condition (2) above. The notion of *tilting dimension* makes sense for objects of $\mathscr{F}(\Delta)'$. Moreover, if we replace every occurrence of $\mathscr{F}(\Delta)$ by $\mathscr{F}(\Delta)'$ in the statement of Corollary 2.11, then the resulting statement is true. An argument by induction on tilting dimension, using the short exact sequence (2.3), shows that $\mathscr{F}(\Delta)' \subset \mathscr{F}(\Delta)$.

Next, let $K^0 \subset \mathrm{K}^{\mathrm{b}}\mathcal{T}$ be the full subcategory consisting of objects isomorphic to a bounded complex of tilting modules $(X^\bullet, d)$ satisfying the following two conditions:

1. The complex is concentrated in nonnegative degrees.
2. The cohomology of the complex vanishes, except possibly in degree 0.

It is easy to see that $\mathscr{F}(\Delta)'$ consists precisely of the objects that lie in the image of K^0 under the functor (2.2). In particular, we see that $\mathscr{F}(\Delta)'$ is stable under extensions, because K^0 is. Thus, to prove that $\mathscr{F}(\Delta) \subset \mathscr{F}(\Delta)'$, it suffices to show that each Δ_γ lies in $\mathscr{F}(\Delta)'$. This follows from the first short exact sequence in Proposition 2.6, by induction on γ. □

The next lemma is ultimately the source of the torsion-freeness in Theorem 1.1.

Lemma 2.12. *Assume that the tilting and cotilting objects in $\mathscr{C}$ coincide. If $X \in \mathscr{F}(\Delta)$, then $\underline{\mathrm{Hom}}(X, \nabla_\gamma)$ is a free module over the graded ring $\underline{\mathrm{End}}(\nabla_\gamma)$.*

Proof. We proceed by induction on the number of steps in a standard filtration of X. If $0 \to X' \to X \to X'' \to 0$ is an exact sequence with $X', X'' \in \mathscr{F}(\Delta)$, then we obtain a short exact sequence

$$0 \to \underline{\mathrm{Hom}}(X'', \nabla_\gamma) \to \underline{\mathrm{Hom}}(X, \nabla_\gamma) \to \underline{\mathrm{Hom}}(X', \nabla_\gamma) \to 0$$

of $\underline{\mathrm{End}}(\nabla_\gamma)$-modules. If the first and last terms are free, the middle term must be as well. Thus, we are reduced to considering the case where X is a standard object, say $X = \Delta_\xi\langle n\rangle$. If $\xi \neq \gamma$, then $\underline{\mathrm{Hom}}(X, \nabla_\gamma) = 0$. If $\xi = \gamma$, then $\underline{\mathrm{Hom}}(X, \nabla_\gamma)$ is a free $\underline{\mathrm{End}}(\nabla_\gamma)$-module by Lemma 2.7(3). □

2.3. **Quotients of the category of tilting objects.** The next result compares the Serre quotient $\mathscr{C}/\mathscr{C}_\Gamma$ to a "naive" quotient category. If $\mathscr{A}$ is an additive category and $\mathscr{B} \subset \mathscr{A}$ is a full subcategory, we write $\mathscr{A} \mathbin{/\!/} \mathscr{B}$ for the category with the same objects as $\mathscr{A}$, but with morphisms given by

$$\mathrm{Hom}_{\mathscr{A}/\!/\mathscr{B}}(X, Y) = \mathrm{Hom}_{\mathscr{A}}(X, Y)/\{f \mid f \text{ factors through an object of } \mathscr{B}\}. \tag{2.4}$$

Proposition 2.13. *Let $\Gamma \subset \Omega$ be a finite order ideal. The quotient functor $\Pi : \mathscr{C} \to \mathscr{C}/\mathscr{C}_\Gamma$ induces an equivalence of categories*

$$\bar{\Pi} : \mathrm{Tilt}(\mathscr{C}) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Gamma) \xrightarrow{\sim} \mathrm{Tilt}(\mathscr{C}/\mathscr{C}_\Gamma). \tag{2.5}$$

Proof. Let $Q : \mathrm{Tilt}(\mathscr{C}) \to \mathrm{Tilt}(\mathscr{C}) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Gamma)$ be the quotient functor. It is clear that $\Pi(\mathrm{Tilt}(\mathscr{C}_\Gamma)) = 0$, so there is a unique functor $\bar{\Pi}$ such that $\bar{\Pi} \circ Q \cong \Pi$. From the classification of tilting objects in Proposition 2.6, it is clear that $\bar{\Pi}$ is essentially surjective. We must prove that it is fully faithful.

We proceed by induction on the size of Γ. Suppose first that Γ is a singleton. Let $T, T' \in \mathrm{Tilt}(\mathscr{C})$, and consider the diagram

$$\begin{tikzcd} \underline{\mathrm{Hom}}_{\mathscr{C}}(T, T') \arrow[r, two heads, "Q"'] \arrow[rr, bend left, "\Pi"] & \underline{\mathrm{Hom}}_{\mathrm{Tilt}(\mathscr{C})/\!/\mathrm{Tilt}(\mathscr{C}_\Gamma)}(T, T') \arrow[r, "\bar{\Pi}"'] & \underline{\mathrm{Hom}}_{\mathscr{C}/\mathscr{C}_\Gamma}(\Pi(T), \Pi(T')) \end{tikzcd} \tag{2.6}$$

By Lemma 2.4, all three terms of the functorial distinguished triangle $\Pi^{\mathrm{L}}\Pi(T) \to T \to \imath^{\mathrm{R}}\imath(T) \to$ lie in $\mathscr{C}$, so that distinguished triangle is actually a short exact sequence. Apply $\underline{\mathrm{Hom}}(-, T')$ to get the long exact sequence

$$\begin{multline} 0 \to \underline{\mathrm{Hom}}(\imath^{\mathrm{L}}(T), \imath^{\mathrm{R}}(T')) \to \underline{\mathrm{Hom}}(T, T') \to \underline{\mathrm{Hom}}(\Pi(T), \Pi(T')) \to \\ \underline{\mathrm{Ext}}^1(\imath^{\mathrm{L}}(T), \imath^{\mathrm{R}}(T')) \to \dots \end{multline} \tag{2.7}$$

The last term vanishes because (by Lemma 2.4 again) $\imath^{\mathrm{L}}(T)$ has a standard filtration and $\imath^{\mathrm{R}}(T')$ has a costandard filtration. It follows that the map labelled Π in (2.6) is surjective, and its kernel can be identified with the space

$$K = \{f : T \to T' \mid f \text{ factors as } T \to \imath\imath^{\mathrm{L}}(T) \to \imath\imath^{\mathrm{R}}(T') \to T'\}.$$

We deduce that $\bar{\Pi}$ is surjective as well. Now, the kernel of Q in (2.6) is the space

$$K' = \{f : T \to T' \mid f \text{ factors through an object of } \mathrm{Tilt}(\mathscr{C}_\Gamma)\}.$$

We already know that $K' \subset K$. But since Γ is a singleton $\{\gamma\}$ with γ necessarily minimal in Ω, we see from Lemma 2.7(1) that $\imath^{\mathrm{L}}(T)$ is actually tilting (and not merely in $\mathscr{F}(\Delta)$), and likewise for $\imath^{\mathrm{R}}(T')$. So $K = K'$, and we conclude that $\bar{\Pi}$ in (2.6) is an isomorphism.

For the general case, choose a nonempty proper ideal $\Upsilon \subset \Gamma$. Then Υ and $\Gamma \setminus \Upsilon$ are both smaller than Γ, and by induction, we have natural equivalences

$$\begin{gathered}
\mathrm{Tilt}(\mathscr{C}) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Upsilon) \cong \mathrm{Tilt}(\mathscr{C}/\mathscr{C}_\Upsilon) \\
\mathrm{Tilt}(\mathscr{C}_\Gamma) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Upsilon) \cong \mathrm{Tilt}(\mathscr{C}_\Gamma/\mathscr{C}_\Upsilon) \\
\mathrm{Tilt}(\mathscr{C}/\mathscr{C}_\Upsilon) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Gamma/\mathscr{C}_\Upsilon) \cong \mathrm{Tilt}((\mathscr{C}/\mathscr{C}_\Upsilon)/(\mathscr{C}_\Gamma/\mathscr{C}_\Upsilon)) \cong \mathrm{Tilt}(\mathscr{C}/\mathscr{C}_\Gamma)
\end{gathered}$$

It is also easy to see that there is a canonical equivalence

$$\mathrm{Tilt}(\mathscr{C}) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Gamma) \cong (\mathrm{Tilt}(\mathscr{C}) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Upsilon)) \mathbin{\big/\!\!\big/} (\mathrm{Tilt}(\mathscr{C}_\Gamma) \mathbin{/\!/} \mathrm{Tilt}(\mathscr{C}_\Upsilon)).$$

Combining all these yields the desired equivalence (2.5). $\square$

The next corollary is immediate from (2.7) and the discussion following it.

Corollary 2.14. *Let $\Gamma \subset \Omega$ be a finite order ideal. If $X \in \mathscr{F}(\Delta)$ and $Y \in \mathscr{F}(\nabla)$, then the map $\underline{\mathrm{Hom}}_{\mathscr{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathscr{C}/\mathscr{C}_\Gamma}(\Pi(X),\Pi(Y))$ is surjective.*

2.4. **Perverse-coherent sheaves on the nilpotent cone.** In this subsection, we assume that $\Bbbk$ is an algebraically closed field of characteristic that is rather good for $G^\vee$. Recall that $\mathcal{N}$ denotes the nilpotent cone of $G^\vee$. Let $G^\vee \times \mathbb{G}_\mathrm{m}$ act on $\mathcal{N}$ by $(g,z) \cdot x = z^{-2}\mathrm{Ad}(g)(x)$. We write $\mathrm{Coh}^{G^\vee \times \mathbb{G}_\mathrm{m}}(\mathcal{N})$, or simply $\mathrm{Coh}(\mathcal{N})$, for the category of $(G^\vee \times \mathbb{G}_\mathrm{m})$-equivariant coherent sheaves on $\mathcal{N}$.

Let $\mathrm{PCoh}(\mathcal{N})$ denote the category of $(G^\vee \times \mathbb{G}_\mathrm{m})$-equivariant *perverse-coherent sheaves* on $\mathcal{N}$. This is the heart of a certain remarkable t-structure on $\mathrm{D^bCoh}(\mathcal{N})$. We refer the reader to [AB, B, A] for details on the definition and properties of this category. Here are some basic facts about $\mathrm{PCoh}(\mathcal{N})$:

- Every object in $\mathrm{PCoh}(\mathcal{N})$ has finite length.
- It is stable under $\mathcal{F} \mapsto \mathcal{F}\langle 1\rangle$, where $\langle 1\rangle : \mathrm{D^bCoh}(\mathcal{N}) \to \mathrm{D^bCoh}(\mathcal{N})$ is given by a twist of the $\mathbb{G}_\mathrm{m}$-action.
- The set $\mathrm{Irr}(\mathrm{PCoh}(\mathcal{N}))/\mathbb{Z}$ is naturally in bijection with $\mathbf{X}^+$.

Remark 2.15. In [A], it is assumed that $\mathrm{char}\,\Bbbk$ is good and that $G^\vee$ has simply connected derived group, but this assumption can be weakened: as long as $\mathrm{char}\,\Bbbk$ is rather good, there is a separable central isogeny $\phi : \tilde{G}^\vee \to G^\vee$ where $\tilde{G}^\vee$ has simply connected derived group, and ϕ induces an equivariant isomorphism of nilpotent varieties. Via ϕ, one can transfer results in [A, Mi] on $\mathrm{PCoh}(\mathcal{N})$ from $\tilde{G}^\vee$ to $G^\vee$.

For $\lambda \in \mathbf{X}^+$, we define a subcategory $\mathrm{PCoh}(\mathcal{N})_{\leq\lambda} \subset \mathrm{PCoh}(\mathcal{N})$ as in (2.1). For our purposes, the most important fact about $\mathrm{PCoh}(\mathcal{N})$ is the following result of Minn-Thu-Aye, which refines the description given in [A, B].

Theorem 2.16 (Minn-Thu-Aye [Mi])**.** *Assume that $\mathrm{char}\,\Bbbk$ is rather good for $G^\vee$. Then the category $\mathrm{PCoh}(\mathcal{N})$ is a graded properly stratified category. Moreover:*

1. *The tilting and cotilting objects in $\mathrm{PCoh}(\mathcal{N})$ coincide, and are given by*
$$T_\lambda = \mathrm{T}(\lambda) \otimes \mathcal{O}_\mathcal{N}.$$
2. *The object $\mathrm{M}(\lambda) \otimes \mathcal{O}_\mathcal{N}$ lies in $\mathrm{PCoh}(\mathcal{N})_{\leq\lambda}$ and has a standard filtration.*
3. *The object $\mathrm{N}(\lambda) \otimes \mathcal{O}_\mathcal{N}$ lies in $\mathrm{PCoh}(\mathcal{N})_{\leq\lambda}$ and has a costandard filtration.*

Proof Sketch. The recollement formalism and the existence of the $\bar{\Delta}_\lambda$ and the $\bar{\nabla}_\lambda$ were established in [A]. (The latter are given by the so-called *Andersen–Jantzen sheaves.*) To show that $\mathrm{PCoh}(\mathcal{N})$ is properly stratified, the main problem is to construct the Δ_λ and the ∇_λ. This is done using the quotient functor

$\Pi : \mathrm{PCoh}(\mathcal{N}) \to \mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_{<\lambda}$ and its adjoints. The last three assertions are proved using criteria like those in [D, Theorem 5], along with the fact that as a $G^\vee$-module, the coordinate ring $\Bbbk[\mathcal{N}]$ has a good filtration [KLT, Theorem 7]. □

Remark 2.17. Both [B] and [A] assert that $\mathrm{PCoh}(\mathcal{N})$ is quasi-hereditary, but those papers use that term in an atypical way, imposing weaker Ext-vanishing conditions on standard objects. It is not quasi-hereditary in the usual sense.

Note that this theorem does *not* say that $\mathrm{M}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ is itself a standard object. Indeed, the standard objects in $\mathrm{PCoh}(\mathcal{N})$ do not, in general, belong to $\mathrm{Coh}(\mathcal{N})$; they are typically complexes in $\mathrm{D^bCoh}(\mathcal{N})$ with cohomology in several degrees. The costandard objects of $\mathrm{PCoh}(\mathcal{N})$ do happen to lie in $\mathrm{Coh}(\mathcal{N})$, but they do not have an elementary description, and they are not generally of the form $\mathrm{N}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$.

Corollary 2.18. *Let $\Gamma \subset \mathbf{X}^+$ be a finite order ideal. Suppose $V_1 \in \mathrm{Rep}(G^\vee)$ has a Weyl filtration, and $V_2 \in \mathrm{Rep}(G^\vee)$ has a good filtration. Then the graded vector space $\underline{\mathrm{Hom}}(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}}))$ is concentrated in even degrees.*

Proof. When $\Gamma = \varnothing$, it is clear that the space $\underline{\mathrm{Hom}}_{\mathrm{Coh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}) \cong \underline{\mathrm{Hom}}_{\mathrm{Rep}(G^\vee)}(V_1, V_2 \otimes \Bbbk[\mathcal{N}])$ is concentrated in even degrees, since the coordinate ring $\Bbbk[\mathcal{N}]$ is concentrated in even degrees. For general Γ, the result then follows from Corollary 2.14. □

3. Background on Parity sheaves

Let X be a complex algebraic variety or ind-variety equipped with a fixed algebraic stratification $X = \bigsqcup_{\gamma \in \Omega} X_\gamma$, where Ω is some indexing set. In the ind-variety case, we assume that the closure of each X_γ is an ordinary finite-dimensional variety; in particular, the closure of each stratum should contain only finitely many other strata. Let $\Bbbk$ be a field. Assume the following conditions hold:

- Each stratum X_γ is simply connected.
- The cohomology groups $H^k(X_\gamma; \Bbbk)$ vanish when k is odd.

Let $\mathrm{D}^{\mathrm{b}}_{\Omega}(X, \Bbbk)$, or simply $\mathrm{D}^{\mathrm{b}}_{\Omega}(X)$, denote the triangulated category of bounded complexes of $\Bbbk$-sheaves on X (in the analytic topology) that are constructible with respect to the given stratification. For each stratum X_γ, let $j_\gamma : X_\gamma \to X$ be the inclusion map.

Definition 3.1. An object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\Omega}(X)$ is said to be $*$-*even* (resp. !-*even*) if for each γ, the cohomology sheaves $\mathcal{H}^k(j_\gamma^* \mathcal{F})$ (resp. $\mathcal{H}^k(j_\gamma^! \mathcal{F})$) vanish for k odd. It is *even* if it is both $*$-even and !-even.

The terms $*$-*odd*, !-*odd*, and *odd* are defined similarly. An object is *parity* if it is a direct sum of an even object and an odd object.

The assumptions above are significantly more restrictive than those in [JMW], but we will not require the full generality of *loc. cit.* The following statement classifies the indecomposable parity objects.

Theorem 3.2 ([JMW, Theorem 2.12])**.** *Let $\mathcal{E}$ be an indecomposable parity object. Then there is a stratum X_γ such that $\mathcal{E}$ is supported on $\overline{X_\gamma}$, and $\mathcal{E}|_{X_\gamma}$ is a shift of the constant sheaf $\underline{\Bbbk}$. Moreover, if $\mathcal{E}'$ is another indecomposable parity object with the same support as $\mathcal{E}$, then $\mathcal{E}'$ is (up to shift) isomorphic to $\mathcal{E}$.*

Definition 3.3. The variety X is said to *have enough parity objects* if for every stratum X_γ, there is an indecomposable parity object $\mathcal{E}_\gamma$ that is supported on the closure $\overline{X_\gamma}$, and such that $\mathcal{E}_\gamma|_{X_\gamma} \cong \underline{\Bbbk}[\dim X_\gamma]$.

For X as above, let $\mathrm{Parity}(X) \subset \mathrm{D}^{\mathrm{b}}_\Omega(X)$ denote the full additive subcategory consisting of parity objects. The main result of this section is the following geometric analogue of Proposition 2.13, comparing a Verdier quotient of $\mathrm{D}^{\mathrm{b}}_\Omega(X)$ to a "naive" quotient (cf. (2.4)). The statement makes use of the following observation: for any closed inclusion of a union of strata $i : Y \to X$, we can identify $\mathrm{Parity}(Y)$ with a full subcategory of $\mathrm{Parity}(X)$ via i_*.

Proposition 3.4. *Assume that X has enough parity objects, and let $Y \subset X$ be a closed union of finitely many strata. The open inclusion $j : X \setminus Y \to X$ induces an equivalence of categories*

$$\bar{j}^* : \mathrm{Parity}(X) \mathbin{/\!/} \mathrm{Parity}(Y) \xrightarrow{\sim} \mathrm{Parity}(X \setminus Y). \tag{3.1}$$

Proof. Let $Q : \mathrm{Parity}(X) \to \mathrm{Parity}(X) \mathbin{/\!/} \mathrm{Parity}(Y)$ be the quotient functor, and let $i : Y \to X$ be the inclusion map. It is clear that $j^*(\mathrm{Parity}(Y)) = 0$, so there is unique functor $\bar{j}^*$ such that $\bar{j}^* \circ Q \cong j^*$. Because X has enough parity objects, the functor $\bar{j}^*$ is essentially surjective. We must prove that it is fully faithful.

We proceed by induction on the number of strata in Y. Suppose first that Y consists of a single closed stratum X_0. Let $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}_\Omega(X)$ be parity objects, and consider the diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathcal{E},\mathcal{F}) \arrow[r, two heads, "Q"'] \arrow[rr, bend left, "j^*"] & \mathrm{Hom}_{\mathrm{Parity}(X)/\!/\mathrm{Parity}(Y)}(\mathcal{E},\mathcal{F}) \arrow[r, "\bar{j}^*"'] & \mathrm{Hom}(j^*\mathcal{E}, j^*\mathcal{F}) \end{tikzcd} \tag{3.2}$$

It suffices to consider the case where $\mathcal{E}$ and $\mathcal{F}$ are both indecomposable. If $\mathcal{E}$ is even and $\mathcal{F}$ is odd, or vice versa, then both $\mathrm{Hom}(\mathcal{E},\mathcal{F})$ and $\mathrm{Hom}(j^*\mathcal{E}, j^*\mathcal{F})$ vanish by [JMW, Corollary 2.8], so $\bar{j}^*$ is trivially an isomorphism. We henceforth assume that $\mathcal{E}$ and $\mathcal{F}$ are both even. (The case where they are both odd is identical.) Apply $\mathrm{Hom}(-,\mathcal{F})$ to the distinguished triangle $j_!j^*\mathcal{E} \to \mathcal{E} \to i_*i^*\mathcal{E} \to$ to get the long exact sequence

$$\cdots \to \mathrm{Hom}(i^*\mathcal{E}, i^!\mathcal{F}) \to \mathrm{Hom}(\mathcal{E},\mathcal{F}) \xrightarrow{j^*} \mathrm{Hom}(j^*\mathcal{E}, j^*\mathcal{F}) \to \mathrm{Hom}^1(i^*\mathcal{E}, i^!\mathcal{F}) \to \cdots$$

Since $i^*\mathcal{E}$ is $*$-even and $i^!\mathcal{F}$ is !-even, we see from [JMW, Corollary 2.8] that the last term above vanishes. It follows that the map labelled j^* in (3.2) is surjective, and its kernel can be identified with the space

$$K = \{ f : \mathcal{E} \to \mathcal{F} \mid f \text{ factors as } \mathcal{E} \to i_*i^*\mathcal{E} \to i_*i^!\mathcal{F} \to \mathcal{F} \} \cong \mathrm{Hom}(i^*\mathcal{E}, i^!\mathcal{F}).$$

We deduce that $\bar{j}^*$ is surjective as well. Now, the kernel of Q in (3.2) is the space

$$K' = \{ f : \mathcal{E} \to \mathcal{F} \mid f \text{ factors through an object of } \mathrm{Parity}(Y) \}.$$

We already know that $K' \subset K$. But since Y consists of a single closed stratum, the object $i^*\mathcal{E}$ is actually even (not just $*$-even), and likewise for $i^!\mathcal{F}$. So $K = K'$, and we conclude that $\bar{j}^*$ in (3.2) is an isomorphism.

For the general case, let S be an open stratum in Y, and let $B = Y \setminus S$ and $X' = X \setminus B$. Then S is closed in X', and by induction, we have natural equivalences

$$\mathrm{Parity}(X') \mathbin{/\!/} \mathrm{Parity}(S) \xrightarrow{\sim} \mathrm{Parity}(X' \setminus S) \cong \mathrm{Parity}(X \setminus Y)$$
$$\mathrm{Parity}(X) \mathbin{/\!/} \mathrm{Parity}(B) \xrightarrow{\sim} \mathrm{Parity}(X \setminus B) \cong \mathrm{Parity}(X')$$
$$\mathrm{Parity}(Y) \mathbin{/\!/} \mathrm{Parity}(B) \xrightarrow{\sim} \mathrm{Parity}(Y \setminus B) \cong \mathrm{Parity}(S)$$

The desired equivalence (3.1) follows from these and the general observation that

$$\mathrm{Parity}(X) \mathbin{/\!/} \mathrm{Parity}(Y) \cong (\mathrm{Parity}(X) \mathbin{/\!/} \mathrm{Parity}(B)) \mathbin{/\!/} (\mathrm{Parity}(Y) \mathbin{/\!/} \mathrm{Parity}(B)). \qquad \square$$

4. Parity sheaves on Kac–Moody flag varieties

In this section, we study Ext-groups of parity sheaves on flag varieties for Kac–Moody groups. The result below will be applied elsewhere in the paper only to affine Grassmannians, but it is no more effort to prove it in this generality. As in the previous section, $\Bbbk$ denotes an arbitrary field.

Theorem 4.1. *Let X be a generalized flag variety for a Kac–Moody group, equipped with the Bruhat stratification, and let $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{(\mathcal{B})}(X)$ be two parity objects with respect to that stratification. The natural map*

$$\mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}_{(\mathcal{B})}(X)}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}^\bullet_{H^\bullet(X;\Bbbk)}(H^\bullet(\mathcal{E}), H^\bullet(\mathcal{F}))$$

is an isomorphism.

For finite flag varieties, this result (with some minor restrictions on $\operatorname{char} \Bbbk$) is due to Soergel [So1, So2]. In [G1], Ginzburg proved a very similar result for simple perverse $\mathbb{C}$-sheaves on smooth projective varieties equipped with a suitable $\mathbb{C}^\times$-action. The proof below follows the outline of Ginzburg's argument quite closely. One exception occurs at a step (see [G1, Proposition 3.2]) where Ginzburg invokes the theory of mixed Hodge modules: here, we substitute an argument of Fiebig–Williamson that relies on the geometry of Schubert varieties.

Remark 4.2. When $\Bbbk = \mathbb{C}$, Ginzburg had already observed in a remark at the end of [G1] that his result could be generalized to the Kac–Moody case. Thus, in that case, this section can be regarded as an exposition of Ginzburg's remark.

We begin with some notation. Let $\mathcal{G}$ be a Kac–Moody group (over $\mathbb{C}$), with maximal torus $\mathcal{T} \subset \mathcal{G}$ and standard Borel subgroup $\mathcal{B} \subset \mathcal{G}$. Let $\mathcal{B}^- \subset \mathcal{G}$ denote the opposite Borel subgroup to $\mathcal{B}$ (with respect to $\mathcal{T}$). Let $\mathcal{P} \subset \mathcal{G}$ be a standard parabolic subgroup of finite type, with Levi factor $L_{\mathcal{P}}$.

For the remainder of the section, X will denote the generalized flag variety $X = \mathcal{G}/\mathcal{P}$, and $\mathrm{D}^{\mathrm{b}}_{(\mathcal{B})}(X)$ will denote the category of complexes of $\Bbbk$-sheaves on X that are constructible with respect to the stratification by $\mathcal{B}$-orbits. (However, certain complexes that are *not* constructible with respect to this stratification will also appear in our arguments.)

Let W (resp. $W_{\mathcal{P}}$) be the Weyl group of $\mathcal{G}$ (resp. $L_{\mathcal{P}}$), and let $W^{\mathcal{P}}$ be the set of minimal-length representatives of the set of cosets $W/W_{\mathcal{P}}$. The length of element $w \in W^{\mathcal{P}}$ will be denoted by $\ell(w)$. It is well known that the $\mathcal{T}$-fixed points and the $\mathcal{B}$-orbits on X are both naturally in bijection with $W^{\mathcal{P}}$. For $w \in W^{\mathcal{P}}$, let $e_w \in X$ be the corresponding $\mathcal{T}$-fixed point, and let $X_w = \mathcal{B} \cdot e_w$ be the corresponding Bruhat cell. We will also need the "opposite Bruhat cell" $X_w^- = \mathcal{B}^- \cdot e_w$. Recall that $X_w \cap X_w^- = \{e_w\}$, and that the intersection is transverse. Moreover, X_w is isomorphic to an affine space of dimension $\ell(w)$. In general, X_w^- may have infinite dimension, but it has codimension $\ell(w)$ (see [K, Lemma 7.3.10]). Let

$$j_w : X_w \to X \qquad \text{and} \qquad \bar{\jmath}_w : X_w^- \to X$$

denote the inclusion maps.

For any closed subset $Z \subset X$, we let $i_Z : Z \to X$ be the inclusion map, and for an object $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}_{(\mathcal{B})}(X)$, we put

$$\mathcal{E}_Z := i_{Z*}i_Z^*\mathcal{E} \qquad \text{and} \qquad \mathcal{E}^Z := i_{Z*}i_Z^!\mathcal{E}.$$

For simplicity, the inclusions of closures of $\mathcal{B}$- and $\mathcal{B}^-$-orbits are denoted

$$i_w : \bar{X}_w \to X \qquad \text{and} \qquad \bar{\imath}_w : \bar{X}_w^- \to X,$$

rather than $i_{\bar{X}_w}$ and $i_{\bar{X}_w^-}$. Recall that $\bar{X}_w$ is known as a *Schubert variety*, and $\bar{X}_w^-$ as a *Birkhoff variety*.

Lemma 4.3. *Let $Z \subset X$ be a finite union of Schubert varieties, and let $X_w \subset Z$ be a Bruhat cell that is open in Z. If $\mathcal{E}$ is $*$-even, then for each k, there is a natural short exact sequence*

$$0 \to H^k(j_{w!}j_w^!\mathcal{E}_Z) \to H^k(\mathcal{E}_Z) \to H^k(\mathcal{E}_{Z \setminus X_w}) \to 0.$$

Proof. The constant map $a : Z \to \{\mathrm{pt}\}$ is a proper, even morphism in the sense of [JMW, Definition 2.33], so by [JMW, Proposition 2.34], if $\mathcal{E}' \in \mathrm{D}^{\mathrm{b}}_{(\mathcal{B})}(Z)$ is $*$-even, then $H^k(\mathcal{E}')$ vanishes when k is odd. All three terms in the distinguished triangle $j_{w!}j_w^!\mathcal{E}_Z \to \mathcal{E}_Z \to \mathcal{E}_{Z\setminus X_w} \to$ are $*$-even, so in the long exact sequence in cohomology, all odd terms vanish, and the even terms give short exact sequences as above. □

Lemma 4.4. *Let $Z \subset X$ be a finite union of Schubert varieties, and let $X_w \subset Z$ be a Bruhat cell that is open in Z. If $\mathcal{E}$ is a parity object, then the natural map $H^k(\mathcal{E}_Z) \to H^k(j_w^*\mathcal{E}_Z)$ is surjective.*

Proof. Let $k_w : \{e_w\} \to X$ be the inclusion map. Since k_w factors through j_w, and j_w factors through i_Z, there is a natural sequence of maps

$$\mathcal{E} \to i_{Z*}i_Z^*\mathcal{E} \to j_{w*}j_w^*\mathcal{E} \to k_{w*}k_w^*\mathcal{E}.$$

Taking cohomology, we obtain a natural sequence of maps

$$H^k(\mathcal{E}) \to H^k(\mathcal{E}_Z) \to H^k(j_w^*\mathcal{E}) \to H^k(k_w^*\mathcal{E}). \tag{4.1}$$

We claim first that the composition $H^k(\mathcal{E}) \to H^k(k_w^*\mathcal{E})$ is surjective. This is essentially the content of [FW, Theorem 5.7(2)]. That result is stated in an abstract, axiomatic setting, but [FW, Proposition 7.1] tells us that it applies to Schubert varieties. Another concern is that [FW, Theorem 5.7(2)] deals with $\mathcal{T}$-equivariant rather than ordinary cohomology. The reader may check that the proof goes through with ordinary cohomology as well. Alternatively, note that both $H^\bullet_{\mathcal{T}}(\mathcal{E})$ and $H^\bullet_{\mathcal{T}}(k_w^*\mathcal{E})$ are free modules over the equivariant cohomology ring of a point $H^\bullet_{\mathcal{T}}(\mathrm{pt})$ by [FW, Proposition 5.6]. In that situation, the ordinary cohomology is obtained from the equivariant cohomology by applying the right-exact functor $\Bbbk \otimes_{H^\bullet_{\mathcal{T}}(\mathrm{pt})} -$. In particular, the surjectivity of $H^\bullet_{\mathcal{T}}(\mathcal{E}) \to H^\bullet_{\mathcal{T}}(k_w^*\mathcal{E})$ implies the surjectivity of the corresponding map in ordinary cohomology.

Next, we claim that the third map in (4.1) is an isomorphism. Since $j_w^*\mathcal{E}$ lies in the triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(X_w)$ generated by the constant sheaf $\underline{\Bbbk}_{X_w}$, it suffices to check that $H^k(\underline{\Bbbk}_{X_w}) \to H^k(\underline{\Bbbk}_{\{e_w\}})$ is an isomorphism. That last claim is obvious.

From these observations, it follows that $H^k(\mathcal{E}_Z) \to H^k(j_w^*\mathcal{E}) \cong H^k(j_w^*\mathcal{E}_Z)$ is surjective as well. □

Lemma 4.5. *There is a canonical isomorphism* $\bar{\jmath}_w^!\underline{\Bbbk}_X \cong \underline{\Bbbk}_{X_w^-}[-2\ell(w)]$.

Proof. Let $\mathcal{U} \subset \mathcal{B}$ be the pro-unipotent radical of the Borel subgroup. For $w \in W^{\mathcal{P}}$, let $\mathcal{U}_w \subset \mathcal{U}$ be the subgroup generated by the root subgroups $\mathcal{U}_\alpha$ where α is a positive root but $w^{-1}\alpha$ is negative. Then $\mathcal{U}_w$ is a finite-dimensional unipotent algebraic group. Let $O_w = \mathcal{U}_w \cdot X_w^- \subset X$. According to [K, Lemma 7.3.10], the multiplication map

$$\mathcal{U}_w \times X_w^- \to O_w$$

is an isomorphism of ind-varieties. Let $i : X_w^- \to \mathcal{U}_w \times X_w^-$ be the obvious inclusion map, and let $i_0 : \{e\} \to \mathcal{U}_w$ be the inclusion of the identity element. Since O_w is open in X, we see that $\bar{\jmath}_w^!\underline{\Bbbk}_X$ is naturally isomorphic to $i^!\underline{\Bbbk}_{\mathcal{U}_w \times X_w^-} \cong i_0^!\underline{\Bbbk}_{\mathcal{U}_w} \boxtimes \underline{\Bbbk}_{X_w^-}$. Since $\mathcal{U}_w$ is isomorphic as a variety to an affine space $\mathbb{A}^{\ell(w)}$, we have a well-known canonical isomorphism $i_0^!\underline{\Bbbk}_{\mathcal{U}_w} \cong \underline{\Bbbk}_{\{e\}}[-2\ell(w)]$, and the result follows. □

Now, let $Y \subset X$ be a finite union of Birkhoff varieties, and let

$$\Lambda_Y = i_Y^!\underline{\Bbbk}_X.$$

Lemma 4.6. *Let* $d_Y = \min\{\ell(w) \mid X_w^- \subset Y\}$. *Then the cohomology sheaves* $\mathcal{H}^i(\Lambda_Y)$ *vanish for* $i < 2d_Y$.

Proof. Let $X_w^- \subset Y$ be such that $\ell(w) = d_Y$. Then X_w^- is necessarily open in Y. Let $A_1 = X_w^-$, and then inductively define $A_2, A_3, \dots$ by setting A_k to be some opposite Bruhat cell X_y^- that is open in $Y \smallsetminus (A_1 \cup \cdots \cup A_{k-1})$. We also let $B_k = A_1 \cup \cdots \cup A_k$. The B_k's form an increasing sequence of open subsets of Y whose union is all of Y. To show that $\mathcal{H}^i(\Lambda_Y)$ vanishes for $i < 2d_Y$, it suffices to show that for all k,

$$\mathcal{H}^i(\Lambda_Y|_{B_k}) = 0 \qquad \text{for } i < 2d_Y. \tag{4.2}$$

We proceed by induction on k. For $k = 1$, we have $\Lambda_Y|_{B_1} \cong \bar{\jmath}_w^!\underline{\Bbbk}_X$, so (4.2) follows from Lemma 4.5. For $k > 1$, let $a : A_k \to B_k$ and $b : B_{k-1} \to B_k$ be the inclusion maps. We have a distinguished triangle

$$a_*a^!(\Lambda_Y|_{B_k}) \to \Lambda_Y|_{B_k} \to b_*b^*(\Lambda|_{B_k}) \to .$$

If $A_k = X_y^-$, the first term can be identified with $a_*\bar{\jmath}_y^!\underline{\Bbbk}_X$. Since $\ell(y) \geq d_Y$, it follows from Lemma 4.5 again that $\mathcal{H}^i(a_*a^!(\Lambda_Y|_{B_k})) = 0$ for $i < 2d_Y$. On the other hand, $b^*(\Lambda|_{B_k}) \cong \Lambda|_{B_{k-1}}$, so $\mathcal{H}^i(a_*b^*(\Lambda_Y|_{B_k}))$ vanishes for $i < 2d_Y$ by induction. Thus, (4.2) holds, as desired. □

Lemma 4.7. *For any* $w \in W^{\mathcal{P}}$, *there is a canonical morphism* $q_w : \underline{\Bbbk}_{\bar{X}_w^-} \to \Lambda_{\bar{X}_w^-}[2\ell(w)]$ *whose restriction to* $X_w^- \subset \bar{X}_w^-$ *is an isomorphism.*

Proof. Let $Y = \bar{X}_w^- \smallsetminus X_w^-$, and let $y : Y \to \bar{X}_w^-$ and $u : X_w^- \to \bar{X}_w^-$ be the inclusion maps. The space Y is a finite union of Birkhoff varieties, and, moreover, the integer d_Y defined in Lemma 4.6 satisfies $d_Y > \ell(w)$. Consider the distinguished triangle

$$y_!\Lambda_Y \to \Lambda_{\bar{X}_w^-} \to u_*u^*\Lambda_{\bar{X}_w^-} \to$$

Note that $u^*\Lambda_{\bar{X}_w^-} \cong \bar{\jmath}_w^! \underline{\Bbbk}_X$. It follows from Lemma 4.5 by adjunction that we have a canonical morphism

$$\underline{\Bbbk}_{\bar{X}_w^-} \to u_* u^* \Lambda_{\bar{X}_w^-}[2\ell(w)]. \tag{4.3}$$

By construction, this map restricts to an isomorphism over X_w^-. Next, we deduce from Lemma 4.6 that

$$\mathrm{Hom}(\underline{\Bbbk}_{\bar{X}_w^-}, y_! \Lambda_Y[2\ell(w)]) = \mathrm{Hom}(\underline{\Bbbk}_{\bar{X}_w^-}, y_! \Lambda_Y[2\ell(w)+1]) = 0.$$

These facts imply that the map in (4.3) factors in a unique way through $\Lambda_{\bar{X}_w^-}$. The resulting map $\underline{\Bbbk}_{\bar{X}_w^-} \to \Lambda_{\bar{X}_w^-}$ is the one we seek. □

Define $c_w : \underline{\Bbbk}_X \to \underline{\Bbbk}_X[2\ell(w)]$ to be the composition

$$\underline{\Bbbk}_X \to \bar{\imath}_{w*} \underline{\Bbbk}_{\bar{X}_w^-} \xrightarrow{\bar{\imath}_{w*} q_w} \bar{\imath}_{w*} \Lambda_{\bar{X}_w^-}[2\ell(w)] \to \underline{\Bbbk}_X[2\ell(w)]. \tag{4.4}$$

We now study the map $\bar{c}_w : \mathrm{Hom}^k(\underline{\Bbbk}, \mathcal{E}_Z) \to \mathrm{Hom}^{k+2\ell(w)}(\underline{\Bbbk}, \mathcal{E}_Z)$ induced by c_w.

Lemma 4.8. *Let Z be a finite union of Schubert varieties, and let $X_w \subset Z$ be a Bruhat cell that is open in Z. For a parity object $\mathcal{E}$, there is a commutative diagram*

$$\begin{tikzcd}
 & & H^k(\mathcal{E}_Z) \arrow[r, two heads] \arrow[d, "\bar{c}_w"] & H^k(j_w^* \mathcal{E}_Z) \arrow[d, "\bar{c}_w", "\wr"'] & \\
0 & H^{k+2\ell(w)}(\mathcal{E}_{Z \smallsetminus X_w}) \arrow[l] & H^{k+2\ell(w)}(\mathcal{E}_Z) \arrow[l, two heads] & H^{k+2\ell(w)}(j_{w!} j_w^! \mathcal{E}_Z) \arrow[l, hook'] & 0 \arrow[l]
\end{tikzcd}$$

Proof. Let $Y = \bar{X}_w \setminus X_w$, and let $y : Y \to X$ be the inclusion map. Since c_w factors through an object supported $\bar{X}_w^-$, and since $\bar{X}_w^- \cap Y = \varnothing$, we see that the composition $H^k(y^! \mathcal{E}_Z) \to H^k(\mathcal{E}_Z) \xrightarrow{\bar{c}_w} H^{k+2\ell(w)}(\mathcal{E}_Z)$ vanishes. In other words, $\bar{c}_w$ must factor through $H^k(\mathcal{E}_Z) \to H^k(j_w^* \mathcal{E}_Z)$. Similar reasoning shows that it must also factor through $H^{k+2\ell(w)}(j_{w!} j_w^! \mathcal{E}_Z) \to H^{k+2\ell(w)}(\mathcal{E}_Z)$, so we at least have a commutative diagram as shown above.

It remains to show that the right-hand vertical map is an isomorphism. Let $p : \{e_w\} \to X_w$ be the inclusion map. Applying i_w^* to (4.4) yields the composition $\underline{\Bbbk}_{X_w} \to p_* \underline{\Bbbk}_{e_w} \to \underline{\Bbbk}_{X_w}[2\ell(w)]$, where the second map comes from adjunction and the identification $\underline{\Bbbk}_{e_w} \cong p^! \underline{\Bbbk}_{X_w}[2\ell(w)]$. Thus, $\bar{c}_w$ is given by the following composition, in which every map is an isomorphism:

$$H^k(j_w^* \mathcal{E}_Z) \to \mathrm{Hom}^k(p_* \underline{\Bbbk}_{e_w}[-2\ell(w)], j_w^* \mathcal{E}_Z) \to \mathrm{Hom}^{k+2\ell(w)}(\underline{\Bbbk}_{e_w}, p^! j_w^! \mathcal{E}_Z) \cong H^{k+2\ell(w)}(j_{w!} j_w^! \mathcal{E}_Z) \qquad \square$$

A very similar argument establishes the following result, whose proof we omit.

Lemma 4.9. *Let Z be a finite union of Schubert varieties, and let $X_w \subset Z$ be a Bruhat cell that is open in Z. For a parity object $\mathcal{F}$, there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & H^k(\mathcal{F}^{Z \smallsetminus X_w}) \arrow[r, hook] & H^k(\mathcal{F}^Z) \arrow[r, two heads] \arrow[d, "\bar{c}_w"] & H^k(j_w^* \mathcal{F}^Z) \arrow[r] \arrow[d, "\bar{c}_w", "\wr"'] & 0 \\
 & & H^{k+2\ell(w)}(\mathcal{F}^Z) & H^{k+2\ell(w)}(j_{w!} j_w^! \mathcal{F}^Z) \arrow[l, hook'] &
\end{tikzcd}$$

With the following proposition, we complete the proof of Theorem 4.1.

Proposition 4.10. *Let $Z \subset X$ be a finite union of Schubert varieties, and let $\mathcal{E}, \mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{(\mathcal{B})}(X)$ be two parity objects. The natural map*

$$\mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}_{(\mathcal{B})}(X)}(\mathcal{E}_Z, \mathcal{F}^Z) \to \mathrm{Hom}^\bullet_{H^\bullet(X;\Bbbk)}(H^\bullet(\mathcal{E}_Z), H^\bullet(\mathcal{F}^Z))$$

is an isomorphism.

Proof Sketch. This is proved by induction on the number of Bruhat cells in Z, via a diagram chase relying on formal consequences of the commutative diagrams in Lemmas 4.8 and 4.9. The argument is essentially identical to the proof of [G1, Proposition 3.10]; see also [G1, Eq. (3.8a–b)]. We omit further details. □

5. Ext-groups of parity sheaves on the affine Grassmannian

In this section, $\Bbbk$ will denote an algebraically closed field whose characteristic is rather good for $G^\vee$. Recall that the $G_{\mathbf{O}}$-orbits are parametrized by $\mathbf{X}^+$. If $\Gamma \subset \mathbf{X}^+$ is a finite order ideal, we can form the closed subset $\mathcal{G}r_\Gamma = \bigcup_{\gamma\in\Gamma} \mathcal{G}r_\gamma$. Let

$$j_\Gamma : U_\Gamma = \mathcal{G}r \setminus \mathcal{G}r_\Gamma \hookrightarrow \mathcal{G}r \tag{5.1}$$

be the complementary open inclusion. For the remainder of the paper, all constructible complexes on $\mathcal{G}r$ or on any subset of $\mathcal{G}r$ should be understood to be constructible with respect to the $G_{\mathbf{O}}$-orbits. In particular, $\mathrm{Parity}(\mathcal{G}r)$ will denote the category of $G_{\mathbf{O}}$-constructible parity objects.

Our goal is compute certain Ext-groups in $\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r)$ or in some $\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(U_\Gamma)$ in terms of $\mathrm{PCoh}(\mathcal{N})$. The main result, a modular generalization of [G2, Proposition 1.10.4], depends on a result of Yun–Zhu [YZ] describing the cohomology of $\mathcal{G}r$. We begin by recalling that result.

Let $\mathsf{e} \in \mathfrak{g}$ be the principal nilpotent element described in [YZ, Proposition 5.6]. Let $B^\vee \subset G^\vee$ be the unique Borel subgroup such that $\mathsf{e} \in \mathrm{Lie}(B^\vee)$, and let $U^\vee \subset B^\vee$ be its unipotent radical. Below, for any subgroup $H \subset G^\vee \times \mathbb{G}_{\mathrm{m}}$, we write H_{e} for the stabilizer of e in H, and we denote by $\mathcal{G}r^\circ$ the identity component of $\mathcal{G}r$.

Proposition 5.1 (Yun–Zhu)**.** *There is a natural isomorphism*

$$H^\bullet(\mathcal{G}r^\circ;\Bbbk) \cong \mathrm{Dist}(U^\vee_{\mathsf{e}}). \tag{5.2}$$

The "naturality" in this proposition refers to a certain compatibility with $\mathcal{S}$. To be more precise, given $M \in \mathrm{Perv}_{G_{\mathbf{O}}}(\mathcal{G}r, \Bbbk)$, the isomorphism (5.2) endows $H^\bullet(M)$ with the structure of a $\mathrm{Dist}(U^\vee_{\mathsf{e}})$-module. On the other hand, if we forget the grading on $H^\bullet(M)$, we obtain the underlying vector space of $\mathcal{S}^{-1}(M) \in \mathrm{Rep}(G^\vee)$. Thus, we can regard $H^\bullet(M)$ as a representation of $U^\vee_{\mathsf{e}} \subset G^\vee$, and hence as a $\mathrm{Dist}(U^\vee_{\mathsf{e}})$-module. In fact, these two $\mathrm{Dist}(U^\vee_{\mathsf{e}})$-module structures on $H^\bullet(M)$ coincide.

Remark 5.2. Proposition 5.1 is stated in [YZ] only when G is quasi-simple and simply connected (in which case $\mathcal{G}r = \mathcal{G}r^\circ$), but it is easily extended to general G by routine arguments. One caveat is that the element e may depend on a choice in general (it is uniquely determined in the quasi-simple case). Once e is fixed, however, the isomorphism (5.2) is still natural in the sense described above.

Let $\mathbb{G}_{\mathrm{m}}$ act on $G^\vee$ by conjugation via the cocharacter $2\rho : \mathbb{G}_{\mathrm{m}} \to T^\vee$, where 2ρ is the sum of the positive roots for G. The resulting semidirect product will be denoted $\mathbb{G}_{\mathrm{m}} \ltimes_{2\rho} G^\vee$. This action preserves the subgroups $G^\vee_{\mathsf{e}}$, $B^\vee_{\mathsf{e}}$, and $U^\vee_{\mathsf{e}}$, so groups such as $\mathbb{G}_{\mathrm{m}} \ltimes_{2\rho} U^\vee_{\mathsf{e}}$ also make sense.

Lemma 5.3. *There are isomorphisms* $(G^\vee \times \mathbb{G}_\mathrm{m})_\mathsf{e} \cong \mathbb{G}_\mathrm{m} \ltimes_{2\rho} B^\vee_\mathsf{e} \cong \mathbb{G}_\mathrm{m} \ltimes_{2\rho} U^\vee_\mathsf{e} \times \mathrm{Z}(G^\vee)$*, where* $\mathrm{Z}(G^\vee)$ *denotes the center of* $G^\vee$*.*

Proof. Let $\phi : G^\vee \times \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m} \ltimes_{2\rho} G^\vee$ be the map $\phi(g, z) = (z, 2\rho(z^{-1})g)$. This map is an isomorphism, and it is easily checked that it takes $(G^\vee \times \mathbb{G}_\mathrm{m})_\mathsf{e}$ to $\mathbb{G}_\mathrm{m} \ltimes_{2\rho} G^\vee_\mathsf{e}$. By [Sp, Theorem 5.9(b)], $G^\vee_\mathsf{e} = B^\vee_\mathsf{e} \cong U^\vee_\mathsf{e} \times \mathrm{Z}(G^\vee)$. □

Lemma 5.4. *Let* $M \in \mathrm{Perv}_{G_\mathbf{O}}(\mathcal{G}r^\circ, \Bbbk)$*. There is a natural isomorphism*

$$\underline{\mathrm{Hom}}_{\mathrm{Coh}(\mathcal{N})}(\mathcal{O}_\mathcal{N}, \mathcal{S}^{-1}(M) \otimes \mathcal{O}_\mathcal{N}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{Dist}(U^\vee_\mathsf{e})}(\Bbbk, H^\bullet(M)).$$

Proof. The assumptions on $\Bbbk$ imply that $\mathcal{N}$ is a normal variety (see, e.g., [J2, Proposition 8.5]). Let $\mathcal{N}_\mathrm{reg} \subset \mathcal{N}$ be the subvariety consisting of regular nilpotent elements, and let $\mathcal{O}_\mathrm{reg}$ denote its structure sheaf. Consider the vector space

$$V = \mathcal{S}^{-1}(M) = H^\bullet(M), \tag{5.3}$$

regarded as an object of $\mathrm{Rep}(G^\vee)$. Since the complement of $\mathcal{N}_\mathrm{reg}$ has codimension at least 2, the restriction map $\Gamma(V \otimes \mathcal{O}_\mathcal{N}) \xrightarrow{\sim} \Gamma(V \otimes \mathcal{O}_\mathrm{reg})$ is an isomorphism of $(G^\vee \times \mathbb{G}_\mathrm{m})$-modules. It follows that

$$\underline{\mathrm{Hom}}_{\mathrm{Coh}(\mathcal{N})}(\mathcal{O}_\mathcal{N}, V \otimes \mathcal{O}_\mathcal{N}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{Coh}^{G^\vee \times \mathbb{G}_\mathrm{m}}(\mathcal{N}_\mathrm{reg})}(\mathcal{O}_\mathrm{reg}, V \otimes \mathcal{O}_\mathrm{reg})$$

is also an isomorphism. Now, $\mathcal{N}_\mathrm{reg}$ is the orbit of the point e under $G^\vee \times \mathbb{G}_\mathrm{m}$, and restriction to e induces an equivalence of categories $\mathrm{Coh}^{G^\vee \times \mathbb{G}_\mathrm{m}}(\mathcal{N}_\mathrm{reg}) \xrightarrow{\sim} \mathrm{Rep}((G^\vee \times \mathbb{G}_\mathrm{m})_\mathsf{e})$. In view of Lemma 5.3, we have a natural isomorphism

$$\underline{\mathrm{Hom}}_{\mathrm{Coh}(\mathcal{N})}(\mathcal{O}_\mathcal{N}, V \otimes \mathcal{O}_\mathcal{N}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{G}_\mathrm{m} \ltimes_{2\rho} U^\vee_\mathsf{e} \times \mathrm{Z}(G^\vee)}(\Bbbk, V),$$

where $\Bbbk$ on the right-hand side denotes the trivial $(\mathbb{G}_\mathrm{m} \ltimes_{2\rho} U^\vee_\mathsf{e} \times \mathrm{Z}(G^\vee))$-module. Since M is supported on $\mathcal{G}r^\circ$, $\mathrm{Z}(G^\vee)$ acts trivially on V, so we may simply omit mentioning it and consider $\mathrm{Hom}_{\mathbb{G}_\mathrm{m} \ltimes_{2\rho} U^\vee_\mathsf{e}}(\Bbbk, V)$.

Now, the category of finite-dimensional $U^\vee_\mathsf{e}$-representations can be identified with a full subcategory of the finite-dimensional $\mathrm{Dist}(U^\vee_\mathsf{e})$-modules [J1, Lemma I.7.16]. Similarly, the category of finite-dimensional $(\mathbb{G}_\mathrm{m} \ltimes_{2\rho} U^\vee_\mathsf{e})$-modules can be identified with a full subcategory of *graded* finite-dimensional $\mathrm{Dist}(U^\vee_\mathsf{e})$-modules, where $\mathrm{Dist}(U^\vee_\mathsf{e})$ itself is graded by the action of $\mathbb{G}_\mathrm{m}$ via the cocharacter 2ρ. This is precisely the grading appearing in (5.2), according to the remarks following [YZ, Theorem 1.1]. On the other hand, the grading on the right-hand side of (5.3) is also given by 2ρ, as seen in [MV2, Theorem 3.6]. Thus, we have

$$\underline{\mathrm{Hom}}_{\mathbb{G}_\mathrm{m} \ltimes_{2\rho} U^\vee_\mathsf{e}}(\Bbbk, V) \cong \underline{\mathrm{Hom}}_{\mathrm{Dist}(U^\vee_\mathsf{e})}(\Bbbk, H^\bullet(M)),$$

and the result follows. □

Lemma 5.5. *Let* $V \in \mathrm{Rep}(G^\vee)$*. The functors*

$$(-) \star \mathcal{S}(V) : \mathrm{D}^\mathrm{b}_{(G_\mathbf{O})}(\mathcal{G}r, \Bbbk) \to \mathrm{D}^\mathrm{b}_{(G_\mathbf{O})}(\mathcal{G}r, \Bbbk),$$
$$(-) \star \mathcal{S}(V^*) : \mathrm{D}^\mathrm{b}_{(G_\mathbf{O})}(\mathcal{G}r, \Bbbk) \to \mathrm{D}^\mathrm{b}_{(G_\mathbf{O})}(\mathcal{G}r, \Bbbk)$$

are adjoint to one another.

Proof. The unit and counit for the adjunction are given by applying $\mathcal{S}$ to the canonical maps $\Bbbk \to V^* \otimes V$ and $V^* \otimes V \to \Bbbk$. □

Proposition 5.6. *For all $V_1, V_2 \in \mathrm{Rep}(G)$, there is a natural map*

$$\mathbb{S} : \mathrm{Hom}^i_{\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r,\Bbbk)}(\mathcal{S}(V_1), \mathcal{S}(V_2)) \to \mathrm{Hom}_{\mathrm{Coh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}\langle i\rangle). \tag{5.4}$$

When $\mathcal{S}(V_1)$ and $\mathcal{S}(V_2)$ are parity sheaves, this is an isomorphism. For any V_1 and V_2, this map is compatible with composition; i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(\mathcal{S}(V_2), \mathcal{S}(V_3)) \otimes \mathrm{Hom}^\bullet(\mathcal{S}(V_1), \mathcal{S}(V_2)) \arrow[r] \arrow[d] & \mathrm{Hom}^\bullet(\mathcal{S}(V_1), \mathcal{S}(V_3)) \arrow[d] \\
\underline{\mathrm{Hom}}(V_2 \otimes \mathcal{O}_{\mathcal{N}}, V_3 \otimes \mathcal{O}_{\mathcal{N}}) \otimes \underline{\mathrm{Hom}}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}) \arrow[r] & \underline{\mathrm{Hom}}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_3 \otimes \mathcal{O}_{\mathcal{N}})
\end{tikzcd} \tag{5.5}$$

Proof. We construct the map (5.4) as the following composition:

$$\begin{aligned}
\mathrm{Hom}^\bullet(\mathcal{S}(V_1), \mathcal{S}(V_2)) &\cong \mathrm{Hom}^\bullet(\mathbf{1}, \mathcal{S}(V_2) \star \mathcal{S}(V_1^*)) && \text{by Lemma 5.5} \\
&\to \underline{\mathrm{Hom}}_{H^\bullet(\mathcal{G}r)}(\Bbbk, H^\bullet(\mathcal{S}(V_2) \star \mathcal{S}(V_1^*))) && \text{taking cohomology} \\
&\cong \underline{\mathrm{Hom}}_{H^\bullet(\mathcal{G}r)}(\Bbbk, H^\bullet(\mathcal{S}(V_2 \otimes V_1^*))) \\
&\cong \underline{\mathrm{Hom}}(\mathcal{O}_{\mathcal{N}}, V_2 \otimes V_1^* \otimes \mathcal{O}_{\mathcal{N}}) && \text{by Lemma 5.4} \\
&\cong \underline{\mathrm{Hom}}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}})
\end{aligned} \tag{5.6}$$

If $\mathcal{S}(V_1)$ and $\mathcal{S}(V_2)$ are parity sheaves, then $\mathcal{S}(V_1^*)$ is as well, and then, by [JMW, Theorem 4.8], so is $\mathcal{S}(V_2) \star \mathcal{S}(V_1^*)$. Theorem 4.1 then tells us that (5.6) is an isomorphism.

Checking the commutativity of (5.5) is mostly an exercise in working with adjunctions; a typical step is checking that the diagram below commutes.

$$\begin{tikzcd}
\Bbbk \otimes \Bbbk = \Bbbk \arrow[rr, "\eta_{V_1}"] \arrow[d, "\eta_{V_1} \otimes \eta_{V_2}"'] & & V_1^* \otimes V_1 \arrow[d, "\mathrm{id} \otimes (g \circ f)"] \\
V_1^* \otimes V_1 \otimes V_2^* \otimes V_2 \arrow[r, "\mathrm{id} \otimes f \otimes \mathrm{id} \otimes g"] & V_1^* \otimes V_2 \otimes V_2^* \otimes V_3 \arrow[r, "\mathrm{id} \otimes \epsilon_{V_2} \otimes \mathrm{id}"] & V_1^* \otimes V_3
\end{tikzcd}$$

Further details are left to the reader. ∎

For the remainder of this section, we will assume that $\mathrm{char}\,\Bbbk$ is also a JMW prime (Definition 1.2) for G. Recall that most good primes are known to be JMW:

Theorem 5.7 ([JMWv1, Theorem 5.1])**.** *Assume G is quasi-simple. If $\mathrm{char}\,\Bbbk$ satisfies the bounds in Table 1, then $\mathcal{S}$ sends every tilting module to a parity sheaf.*

Proposition 5.8. *There is an equivalence of additive categories*

$$\mathbb{S} : \mathrm{Parity}(\mathcal{G}r) \to \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N}))$$

such that for any tilting G-module V, we have $\mathbb{S}(\mathcal{S}(V)[n]) \cong V \otimes \mathcal{O}_{\mathcal{N}}\langle n\rangle$.

Proof. Every indecomposable object in $\mathrm{Parity}(\mathcal{G}r)$ is isomorphic to some $\mathcal{S}(V)[n]$, where $V \in \mathrm{Rep}(G)$ is a tilting module. Similarly, every indecomposable tilting object in $\mathrm{PCoh}(\mathcal{N})$ is of the form $V \otimes \mathcal{O}_{\mathcal{N}}\langle n\rangle$ for such a V. Thus, Proposition 5.6 implies that the full subcategory of indecomposable objects in $\mathrm{Parity}(\mathcal{G}r)$ is equivalent to the full subcategory of indecomposable objects in $\mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N}))$. Such an equivalence extends in a unique way (up to isomorphism) to an equivalence $\mathrm{Parity}(\mathcal{G}r) \xrightarrow{\sim} \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N}))$. ∎

Corollary 5.9. *Let $\Gamma \subset \mathbf{X}^+$ be a finite order ideal. There is an equivalence of categories $\mathbb{S}_\Gamma$, unique up to isomorphism, that makes the following diagram commute up to isomorphism:*

$$\begin{tikzcd}
\mathrm{Parity}(\mathcal{G}r) \arrow[r, "\mathbb{S}"] \arrow[d, "j_\Gamma^*"'] & \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N})) \arrow[d, "\Pi_\Gamma"] \\
\mathrm{Parity}(U_\Gamma) \arrow[r, "\mathbb{S}_\Gamma"] & \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_\Gamma)
\end{tikzcd}$$

Proof. The functor $\mathbb{S}$ of Proposition 5.8 restricts to an equivalence $\mathrm{Parity}(\mathcal{G}r_\Gamma) \xrightarrow{\sim} \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N})_\Gamma)$, and so it induces an equivalence of quotient categories

$$\mathrm{Parity}(\mathcal{G}r) \mathbin{/\!/} \mathrm{Parity}(\mathcal{G}r_\Gamma) \xrightarrow{\sim} \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N})) \mathbin{/\!/} \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N})_\Gamma).$$

Propositions 2.13 and 3.4 then give us the result. □

Theorem 5.10. *Let $\Gamma \subset \mathbf{X}^+$ be a finite order ideal. If V_1 has a Weyl filtration and V_2 a good filtration, there is a natural isomorphism of graded vector spaces*

$$\hat{\mathbb{S}}_\Gamma : \mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(U_\Gamma)}(S(V_1)|_{U_\Gamma}, S(V_2)|_{U_\Gamma}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_\Gamma}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N})). \tag{5.7}$$

This map is compatible with (5.4)*, in the sense that the diagram*

$$\begin{tikzcd}
\mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r)}(\mathcal{S}(V_1), \mathcal{S}(V_2)) \arrow[r, "\mathbb{S}", "\sim"'] \arrow[d, "j_\Gamma^*"'] & \underline{\mathrm{Hom}}_{\mathrm{PCoh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_\mathcal{N}, V_2 \otimes \mathcal{O}_\mathcal{N}) \arrow[d, "\Pi_\Gamma"] \\
\mathrm{Hom}^\bullet_{\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(U_\Gamma)}(\mathcal{S}(V_1)|_{U_\Gamma}, \mathcal{S}(V_2)|_{U_\Gamma}) \arrow[r, "\hat{\mathbb{S}}_\Gamma", "\sim"'] & \underline{\mathrm{Hom}}_{\frac{\mathrm{PCoh}(\mathcal{N})}{\mathrm{PCoh}(\mathcal{N})_\Gamma}}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N}))
\end{tikzcd} \tag{5.8}$$

commutes. Moreover, (5.7) *is compatible with composition: if V_1 has a Weyl filtration, V_2 is tilting, and V_3 has a good filtration, then the following diagram commutes:*

$$\begin{tikzcd}
\begin{matrix}\mathrm{Hom}^\bullet(\mathcal{S}(V_2)|_{U_\Gamma}, \mathcal{S}(V_3)|_{U_\Gamma}) \otimes \\ \mathrm{Hom}^\bullet(\mathcal{S}(V_1)|_{U_\Gamma}, \mathcal{S}(V_2)|_{U_\Gamma})\end{matrix} \arrow[r] \arrow[d] & \mathrm{Hom}^\bullet(\mathcal{S}(V_1)|_{U_\Gamma}, \mathcal{S}(V_3)|_{U_\Gamma}) \arrow[d] \\
\begin{matrix}\underline{\mathrm{Hom}}(\Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_3 \otimes \mathcal{O}_\mathcal{N})) \otimes \\ \underline{\mathrm{Hom}}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N}))\end{matrix} \arrow[r] & \underline{\mathrm{Hom}}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_3 \otimes \mathcal{O}_\mathcal{N}))
\end{tikzcd} \tag{5.9}$$

Note that, in contrast with (5.4), the map $\hat{\mathbb{S}}_\Gamma$ is only defined when V_1 has a Weyl filtration and V_2 a good filtration, and not for general objects of $\mathrm{Rep}(G)$.

Proof. For brevity, we will write $\hat{\mathbb{S}}$ for $\hat{\mathbb{S}}_\Gamma$, U for U_Γ, and Π for Π_Γ throughout the proof. We proceed by induction on the tilting dimension of V_1 and V_2.

If V_1 and V_2 both have tilting dimension 0, i.e., if they are both tilting, then we simply take $\hat{\mathbb{S}}$ to be the map induced by the functor $\mathbb{S}_\Gamma$ from Corollary 5.9. The commutativity of both (5.8) and (5.9) is immediate from that proposition.

Suppose now that the result is known when V_1 has tilting dimension $\leq n_1$ and V_2 has tilting dimension $\leq n_2$. Now, let $V_1 \in \mathrm{Rep}(G^\vee)$ have a Weyl filtration and tilting dimension $n_1 + 1$. By Corollary 2.11, we can find a short exact sequence

$$0 \to V_1 \to T \to V_1' \to 0 \tag{5.10}$$

$$
\begin{tikzcd}
\mathrm{Hom}^{i-1}(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) = 0 \arrow[d] & \\
\mathrm{Hom}^{i-1}(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) \arrow[d] & 0 \arrow[d] \\
\mathrm{Hom}^{i}(\mathcal{S}(V_1')|_U, \mathcal{S}(V_2)|_U) \arrow[r, "\hat{\mathbb{S}}", "\sim"'] \arrow[d, "u"'] \arrow[dr, phantom, "(*)"] & \mathrm{Hom}(\Pi(V_1' \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) \arrow[d, "u'"] \\
\mathrm{Hom}^{i}(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) \arrow[r, "\hat{\mathbb{S}}", "\sim"'] \arrow[d, "v"'] & \mathrm{Hom}(\Pi(T \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) \arrow[d, "v'"] \\
\mathrm{Hom}^{i}(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) \arrow[r, dashed] \arrow[d] & \mathrm{Hom}(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) \arrow[d] \\
\mathrm{Hom}^{i+1}(\mathcal{S}(V_1')|_U, \mathcal{S}(V_2)|_U) \arrow[r, "\hat{\mathbb{S}}", "\sim"'] & 0
\end{tikzcd}
$$

FIGURE 1. Commutative diagram for the proof of Theorem 5.10

where T is tilting and V_1' has a Weyl filtration and tilting dimension n_1.

Let V_2 have a good filtration and tilting dimension $\leq n_2$. Let i be an even integer, and consider the commutative diagram in Figure 1. The left-hand column is the long exact sequence induced by (5.10). The right-hand column is also induced by (5.10). It is a short exact sequence because, by Theorem 2.16 and Lemma 2.3, $\Pi(V_1' \otimes \mathcal{O}_{\mathcal{N}})$ and $\Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})$ have standard and costandard filtrations, respectively, in $\mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_\Gamma$, and so $\underline{\mathrm{Ext}}^1(\Pi(V_1' \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})) = 0$.

By induction, we have $\mathrm{Hom}^\bullet(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) \cong \underline{\mathrm{Hom}}(\Pi(T \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}}))$. In particular, since i is even, we have $\mathrm{Hom}^{i-1}(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) = 0$ by Corollary 2.18. Next, because the square $(*)$ involving u and u' commutes, the map u must be injective. It follows that

$$\mathrm{Hom}^{i-1}(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) = 0 \qquad \text{for } i-1 \text{ odd.}$$

The left-hand column of Figure 1 has now been reduced to a short exact sequence. It is clear that there is a unique isomorphism

$$\hat{\mathbb{S}} : \mathrm{Hom}^i(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) \xrightarrow{\sim} \mathrm{Hom}(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) \tag{5.11}$$

that makes Figure 1 commute. For now, the map we have constructed appears to depend on the choice of (5.10). We will address this issue later.

First, let us consider the special case where $\Gamma = \varnothing$, so that $U = \mathcal{G}r$, and Π is the identity functor. In this case, the solid horizontal arrows in Figure 1 are given by (5.4), by induction. Since the dotted arrow is uniquely determined, it too must be given by (5.4).

Now, compare the special case ($\Gamma = \varnothing$) of Figure 1 with the general case. Since (5.8) holds for the pairs (V_1', V_2) and (T, V_2) by induction, one can see by an easy diagram chase that it also holds for the pair (V_1, V_2).

Recall from Corollary 2.14 that the right-hand vertical map in (5.8) is surjective. Since the horizontal maps are isomorphisms, the left-hand vertical map must be surjective as well. Once we know that both vertical maps are surjective, we can see that the bottom horizontal map is uniquely determined. Thus, the map (5.11) is indepenedent of (5.10).

It remains to show that (5.11) is natural in both variables, and that (5.9) commutes. The former is essentially a special case of the latter, so we focus on the latter. In the special case $\Gamma = \varnothing$, the commutativity of (5.9) is contained in Proposition 5.6. For general Γ, we deduce the result by a diagram chase using the special case $\Gamma = \varnothing$ together with several instances of (5.8).

An entirely similar argument establishes the required induction step involving the tilting dimension of V_2. □

6. Proof of the Mirković–Vilonen conjecture

We begin with a lemma about sheaves on a single $G_{\mathbf{O}}$-orbit $\mathcal{G}r_\lambda$ in $\mathcal{G}r$. For this lemma, $\Bbbk$ may be any field. Note that $\mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r_\lambda)$ is the category of complexes of sheaves whose cohomology sheaves are (locally) constant.

Lemma 6.1. *The following conditions on an object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{(G_{\mathbf{O}})}(\mathcal{G}r_\lambda)$ are equivalent:*

(1) *$\mathcal{F}$ is even.*
(2) *$\mathrm{Hom}^\bullet(\mathcal{F}, \underline{\Bbbk})$ is a free $H^\bullet(\mathcal{G}r_\lambda)$-module generated in even degrees.*

Proof. Every even object is a direct sum of objects of the form $\underline{\Bbbk}[2n]$, so it is clear that the first condition implies the second. Suppose now that the second condition holds. Choose a basis $e_1, \ldots, e_k$ for $\mathrm{Hom}^\bullet(\mathcal{F}, \Bbbk)$ as a $H^\bullet(\mathcal{G}r_\lambda)$-module, and suppose each e_i is homogeneous of degree $2n_i$. That is, each e_i is a morphism $\mathcal{F} \to \underline{\Bbbk}[2n_i]$. Let $\mathcal{F}' = \bigoplus_{i=1}^k \underline{\Bbbk}[2n_i]$, and consider the map $f = (e_1, \ldots, e_k) : \mathcal{F} \to \mathcal{F}'$. The map $\mathrm{Hom}^\bullet(\mathcal{F}', \underline{\Bbbk}) \to \mathrm{Hom}^\bullet(\mathcal{F}, \underline{\Bbbk})$ induced by f is surjective (all the e_i lie in its image), and hence, since these are finite-dimensional vector spaces, it is an isomorphism. Therefore, letting $\mathcal{G}$ denote the cone of $f : \mathcal{F} \to \mathcal{F}'$, we have

$$\mathrm{Hom}^\bullet(\mathcal{G}, \Bbbk) = 0. \tag{6.1}$$

We claim that $\mathcal{G} = 0$. If not, let n be the top degree in which $H^n(\mathcal{G}) \neq 0$. Then, there is a nonzero truncation morphism $\mathcal{G} \to \tau_{\geq n}\mathcal{G} \cong H^n(\mathcal{G})[-n]$. The constant sheaf $H^n(\mathcal{G})$ is a direct sum of copies of $\underline{\Bbbk}$, so there is a nonzero map $\mathcal{G} \to \underline{\Bbbk}[-n]$, contradicting (6.1). Thus, $\mathcal{G} = 0$, and f is an isomorphism. In particular, $\mathcal{F} \cong \bigoplus \underline{\Bbbk}[2n_i]$ is even. □

Theorem 6.2 ([JMWv1, Conjecture 5.5])**.** *Assume that* $\operatorname{char} \Bbbk$ *is a JMW prime for G. Then the perverse sheaf $\mathcal{I}_!(\lambda)$ is $*$-parity.*

More precisely, $\mathcal{I}_!(\lambda)$ is $*$-even (resp. $*$-odd) if $\dim \mathcal{G}r_\lambda$ is even (resp. odd).

Proof. It is well known that every component of $\mathcal{G}r$ is isomorphic to a component of the affine Grassmannian for the group $G/\mathrm{Z}(G)$, via an isomorphism compatible with the stratification by $G_{\mathbf{O}}$-orbits. Thus, we may assume that G is semisimple and of adjoint type. In that case, $G^\vee$ is semisimple and simply connected, so we can invoke the results of Section 5 without further restrictions on $\operatorname{char} \Bbbk$.

For simplicity, let us assume that $\dim \mathcal{G}r_\lambda$ is even; the argument in the odd case is the same. Let $\mu \in \mathbf{X}^+$ be a weight such that $\mathcal{G}r_\mu \subset \overline{\mathcal{G}r_\lambda}$. Recall that this implies that $\dim \mathcal{G}r_\mu$ is also even.

Let $\Gamma \subset \mathbf{X}^+$ be the set of weights that are strictly smaller than μ. Let $U = U_\Gamma = \mathcal{G}r \smallsetminus \mathcal{G}r_\Gamma$, and let $j = j_\Gamma$ as in (5.1). By Theorem 5.10, we have a natural isomorphism

$$\mathrm{Hom}^\bullet(j^*\mathcal{I}_!(\lambda), j^*\mathcal{T}(\mu)) \cong \underline{\mathrm{Hom}}(\Pi(\mathrm{M}(\lambda)\otimes\mathcal{O}_{\mathcal{N}}), \Pi(\mathrm{T}(\mu)\otimes\mathcal{O}_{\mathcal{N}})).$$

In particular, by (5.9), this is an isomorphism of graded modules over the ring

$$\mathrm{Hom}^\bullet(j^*\mathcal{T}(\mu), j^*\mathcal{T}(\mu)) \cong \underline{\mathrm{End}}(\Pi(\mathrm{T}(\mu)\otimes\mathcal{O}_{\mathcal{N}})).$$

Finally, in the quotient category $\mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_\Gamma$, the tilting object $\Pi(\mathrm{T}(\mu)\otimes\mathcal{O}_{\mathcal{N}})$ coincides with the costandard object $\Pi(\mathrm{N}(\mu)\otimes\mathcal{O}_{\mathcal{N}})$, by Lemma 2.7(1). By Lemma 2.12, the space $\underline{\mathrm{Hom}}(\Pi(\mathrm{M}(\lambda)\otimes\mathcal{O}_{\mathcal{N}}), \Pi(\mathrm{N}(\mu)\otimes\mathcal{O}_{\mathcal{N}}))$ is a free $\underline{\mathrm{End}}(\Pi(\mathrm{N}(\mu)\otimes\mathcal{O}_{\mathcal{N}}))$-module, and by Corollary 2.18, it is generated in even degrees.

Now, let $i : \mathcal{G}r_\mu \to U$ be the inclusion map. This is a closed inclusion, and we clearly have $j^*\mathcal{T}(\mu) \cong i_*\underline{\Bbbk}[2d]$, where $2d = \dim \mathcal{G}r_\mu$. Rephrasing the conclusion of the previous paragraph, we have that

$$\mathrm{Hom}^\bullet(j^*\mathcal{I}_!(\lambda), i_*\underline{\Bbbk}_{\mathcal{G}r_\lambda}) \cong \mathrm{Hom}^\bullet(i^*j^*\mathcal{I}_!(\lambda), \underline{\Bbbk})$$

is a free module generated in even degrees over the ring

$$\mathrm{Hom}^\bullet(i_*\underline{\Bbbk}_{\mathcal{G}r_\mu}, i_*\underline{\Bbbk}_{\mathcal{G}r_\mu}) \cong H^\bullet(\mathcal{G}r_\mu).$$

By Lemma 6.1, $i^*j^*\mathcal{I}_!(\lambda)$ is even, as desired. □

Theorem 6.3 (cf. [MV1, Conjecture 6.3] or [MV2, Conjecture 13.3])**.** *If p is a JMW prime for G, then the stalks of $\mathcal{I}_!(\lambda,\mathbb{Z})$ have no p-torsion.*

Proof. Let M be a $\mathbb{Z}$-module. It is a routine exercise to show that if M has p-torsion, then $H^i(M \otimes^L \bar{\mathbb{F}}_p) \neq 0$ for both $i = 0$ and $i = -1$. Now, let $x \in \mathcal{G}r$, and consider the stalk $\mathcal{I}_!(\lambda,\mathbb{Z})_x$, which is an object in the derived category of $\mathbb{Z}$-modules. Since $\mathbb{Z}$ has global dimension 1, $\mathcal{I}_!(\lambda,\mathbb{Z})_x$ is isomorphic to the direct sum of its cohomology modules, and if any cohomology module had p-torsion, the object $\mathcal{I}_!(\lambda,\mathbb{Z})_x \otimes^L \bar{\mathbb{F}}_p$ would have nonzero cohomology in both even and odd degrees. But by [MV2, Proposition 8.1(a)], we have

$$\mathcal{I}_!(\lambda,\mathbb{Z})_x \overset{L}{\otimes} \bar{\mathbb{F}}_p \cong \mathcal{I}_!(\lambda, \bar{\mathbb{F}}_p)_x,$$

and Theorem 6.2 tells us that the latter cannot have cohomology in both even and odd degrees. Thus, $\mathcal{I}_!(\lambda,\mathbb{Z})_x$ has no p-torsion. □

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