STAGGERED SHEAVES ON PARTIAL FLAG VARIETIES

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Abstract. *Staggered t-structures* are a class of t-structures on derived categories of equivariant coherent sheaves. In this note, we show that the derived category of coherent sheaves on a partial flag variety, equivariant for a Borel subgroup, admits an artinian staggered t-structure. As a consequence, we obtain a basis for its equivariant K-theory consisting of simple staggered sheaves.

Let X be a variety over an algebraically closed field, and let G be an algebraic group acting on X with finitely many orbits. Let $\mathfrak{Coh}^G(X)$ be the category of G-equivariant coherent sheaves on X, and let $\mathcal{D}^G(X)$ denote its bounded derived category. *Staggered sheaves*, introduced in [1], are the objects in the heart of a certain t-structure on $\mathcal{D}^G(X)$, generalizing the perverse coherent t-structure [2]. The definition of this t-structure depends on the following data: (1) an *s-structure* on X (see below); (2) a choice of a Serre–Grothendieck dualizing complex $\omega_X \in \mathcal{D}^G(X)$ [4]; and (3) a *perversity*, which is an integer-valued function on the set of G-orbits, subject to certain constraints. When the perversity is "strictly monotone and comonotone," the category of staggered sheaves is particularly nice: every object has finite length, and every simple object arises by applying an intermediate-extension ("IC") functor to an irreducible vector bundle on a G-orbit.

An s-structure on X is a collection of full subcategories $(\{\mathfrak{Coh}^G(X)_{\le n}\}, \{\mathfrak{Coh}^G(X)_{\ge n}\})_{n\in\mathbb{Z}}$, satisfying various conditions involving Hom- and Ext-groups, tensor products, and short exact sequences. The *staggered codimension* of the closure of an orbit $i_C : C \to X$, denoted $\operatorname{scod} \overline{C}$, is defined to be $\operatorname{codim} \overline{C} + n$, where n is the unique integer such that $i_C^! \omega_X \in \mathcal{D}^G(C)$ is a shift of an object in $\mathfrak{Coh}^G(C)_{\le n} \cap \mathfrak{Coh}^G(C)_{\ge n}$. By [1, Theorem 9.9], a sufficient condition for the existence of a strictly monotone and comonotone perversity is that staggered codimensions of neighboring orbits differ by at least 2. The goal of this note is to establish the existence of a well-behaved staggered category on partial flag varieties, by constructing an s-structure and computing staggered codimensions. As a consequence, we obtain a basis for the equivariant K-theory $K^B(G/P)$ consisting of simple staggered sheaves.

1. A gluing theorem for s-structures

If X happens to be a single G-orbit, s-structures on X can be described via the equivalence between $\mathfrak{Coh}^G(X)$ and the category of finite-dimensional representations of the isotropy group of X. In the general case, however, specifying an s-structure on X directly can be quite arduous. The following "gluing theorem" lets us specify an s-structure on X by specifying one on each G-orbit.

Theorem 1.1. *For each orbit $C \subset X$, let $\mathcal{I}_C \subset \mathcal{O}_X$ denote the ideal sheaf corresponding to the closed subscheme $i_C : \overline{C} \hookrightarrow X$. Suppose each orbit C is endowed with an s-structure, and that $i_C^* \mathcal{I}_C|_C \in \mathfrak{Coh}^G(C)_{\le -1}$. There is a unique s-structure on X whose restriction to each orbit is the given s-structure.*

Proof. This statement is nearly identical to [1, Theorem 10.2]. In that result, the requirement that $i_C^* \mathcal{I}_C|_C \in \mathfrak{Coh}^G(C)_{\le -1}$ is replaced by the following two assumptions:

(F1) For each orbit C, $i_C^* \mathcal{I}_C|_C \in \mathfrak{Coh}^G(C)_{\le 0}$.

(F2) Each $\mathcal{F} \in \mathfrak{Coh}^G(C)_{\le w}$ admits an extension $\mathcal{F}_1 \in \mathfrak{Coh}^G(\overline{C})$ whose restriction to any smaller orbit $C' \subset \overline{C}$ is in $\mathfrak{Coh}^G(C')_{\le w}$.

Condition (F1) is trivially implied by the stronger assumption that $i_C^* \mathcal{I}_C|_C \in \mathfrak{Coh}^G(C)_{\le -1}$. It suffices, then, to show that (F2) is implied by it as well. Given $\mathcal{F} \in \mathfrak{Coh}^G(C)_{\le w}$, let $\mathcal{G} \in \mathfrak{Coh}^G(\overline{C})$ be some sheaf such that $\mathcal{G}|_C \simeq \mathcal{F}$. Let $C' \subset \overline{C} \smallsetminus C$ be a maximal orbit (with respect to the closure partial order) such that $i_{C'}^* \mathcal{G}|_{C'} \notin \mathfrak{Coh}^G(C')_{\le w}$. (If there is no such C', then $\mathcal{G}$ is the desired extension of $\mathcal{F}$, and there is nothing to prove.) Let $v \in \mathbb{Z}$ be such that $i_{C'}^* \mathcal{G}|_{C'} \in \mathfrak{Coh}^G(C')_{\le v}$. By assumption, we have $v > w$. Let $\mathcal{G}' = \mathcal{G} \otimes \mathcal{I}_{C'}^{\otimes v-w}$. Since $\mathcal{I}_{C'}|_{X \smallsetminus \overline{C}'}$ is isomorphic to the structure sheaf of $X \smallsetminus \overline{C}'$,

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we see that $\mathcal{G}'|_{\overline{C}\smallsetminus\overline{C'}} \simeq \mathcal{G}|_{\overline{C}\smallsetminus\overline{C'}}$. On the other hand, according to [1, Axiom (S6)] (which describes how tensor products behave with respect to s-structures), the fact that $i^*_{C'}\mathcal{I}_{C'}|_{C'} \in \mathfrak{Coh}^G(C')_{\le -1}$ implies that $i^*_{C'}\mathcal{G}'|_{C'} \simeq i^*_{C'}\mathcal{G}|_{C'} \otimes (i^*_{C'}\mathcal{I}_{C'}|_{C'})^{\otimes v-w} \in \mathfrak{Coh}^G(C')_{\le w}$. Thus, $\mathcal{G}'$ is a new extension of $\mathcal{F}$ such that the number of orbits in $\overline{C}\smallsetminus C$ where (F2) fails is fewer than for $\mathcal{G}$. Since the total number of orbits is finite, this construction can be repeated until an extension $\mathcal{F}_1$ satisfying (F2) is obtained. □

2. Torus actions on affine spaces

In this section, we consider coherent sheaves on an affine space. Let T be an algebraic torus over an algebraically closed field k, and let Λ be its weight lattice. Choose a set of weights $\lambda_1,\dots,\lambda_n \in \Lambda$. Let T act linearly on $\mathbb{A}^n = \operatorname{Spec} k[x_1,\dots,x_n]$ by having it act with weight λ_i on the line defined by the ideal $(x_j : j \ne i)$. Given $\mu \in \Lambda$, let $V(\mu)$ denote the one-dimensional T-representation of weight μ. If X is an affine space with a T-action, we denote by $\mathcal{O}_X(\mu)$ the twist of the structure sheaf of X by μ.

Suppose $m \le n$, and identify $\mathbb{A}^m$ with the closed subspace of $\mathbb{A}^n$ defined by the ideal $(x_j : j > m)$. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{A}^n}$ denote the corresponding ideal sheaf, and let $i : \mathbb{A}^m \hookrightarrow \mathbb{A}^n$ be the inclusion map.

Proposition 2.1. *With the above notation, we have*

$$i^*\mathcal{I} \simeq \mathcal{O}_{\mathbb{A}^m}(-\lambda_{m+1}) \oplus \cdots \oplus \mathcal{O}_{\mathbb{A}^m}(-\lambda_n) \qquad \text{and} \qquad i^!\mathcal{O}_{\mathbb{A}^n}(\mu) \simeq \mathcal{O}_{\mathbb{A}^m}(\mu + \lambda_{m+1} + \cdots \lambda_n)[m-n].$$

Proof. Throughout, we will pass freely between coherent sheaves and modules, and between ideal sheaves and ideals. In the T-action on the ring $R = k[x_1,\dots,x_n]$, T acts on the one-dimensional space kx_i with weight $-\lambda_i$. We have $i^*\mathcal{I} \simeq \mathcal{I}/\mathcal{I}^2 \simeq (x_{m+1},\dots,x_n)/(x_ix_j : m+1 \le i < j \le n)$, so if we let $S = k[x_1,\dots,x_m]$, we obtain $i^*\mathcal{I} \simeq x_{m+1}S \oplus \cdots \oplus x_nS \simeq V(-\lambda_{m+1}) \otimes S \oplus \cdots \oplus V(-\lambda_n) \otimes S$.

To calculate $i^!\mathcal{O}_{\mathbb{A}^n}(\mu)$, we may assume that $m = n-1$, as the general case then follows by induction. Recall that $i_*i^!(\cdot) \simeq R\mathcal{H}om(i_*\mathcal{O}_{\mathbb{A}^{n-1}},\cdot)$. To compute the latter functor, we employ the projective resolution $x_nR \hookrightarrow R$ for $i_*\mathcal{O}_{\mathbb{A}^{n-1}}$. Now, $x_nR \simeq V(-\lambda_n) \otimes R$, so when we apply $\operatorname{Hom}(\cdot, V(\mu)\otimes R)$ to this sequence, we obtain an injective map $V(\mu) \otimes R \to V(\mu+\lambda_n) \otimes R$ whose image is $V(\mu+\lambda_n) \otimes x_nR$. The cohomology of this complex vanishes except in degree 1, where we find $V(\mu+\lambda_n) \otimes R/x_nR$. Thus, $i_*i^!\mathcal{O}_{\mathbb{A}^n}(\mu) \simeq R\mathcal{H}om(i_*\mathcal{O}_{\mathbb{A}^{n-1}}, \mathcal{O}_{\mathbb{A}^n}(\mu)) \simeq i_*\mathcal{O}_{\mathbb{A}^{n-1}}(\mu+\lambda_n)[-1]$, as desired. □

3. s-structures on Bruhat cells

Let G be a reductive algebraic group over an algebraically closed field, and let $T \subset B \subset P$ be a maximal torus, a Borel subgroup, and a parabolic subgroup, respectively, and let L be the Levi subgroup of P.

Let W be the Weyl group of G (with respect to T), and let Φ be its root system. Let Φ^+ be the set of positive roots corresponding to B. Let $W_L \subset W$ and $\Phi_L \subset \Phi$ be the Weyl group and root system of L, and let $\Phi_P = \Phi_L \cup \Phi^+$. For each $w \in W$, we fix once and for all a representative in G, also denoted w. Let X°_w denote the Bruhat cell BwP/P, let X_w denote its closure (a Schubert variety), and let $i_w : X_w \to G/P$ be the inclusion. Note that $X^\circ_w = X^\circ_v$ if and only if $wW_L = vW_L$.

Let Λ denote the weight lattice of T, and let $\rho = \frac{1}{2}\sum\Phi^+$. (For a set $\Psi \subset \Phi$, we write "$\sum\Psi$" for $\sum_{\alpha\in\Psi}\alpha$.) For any $w \in W$, we define various subsets of Φ^+ and elements of Λ as follows:

$$\begin{aligned} &\Pi(w) = \Phi^+ \cap w(\Phi^+) && \pi(w) = \textstyle\sum\Pi(w) && \Pi_L(w) = \Phi^+ \cap w(\Phi^+ \smallsetminus \Phi_L) && \pi_L(w) = \textstyle\sum\Pi_L(w) \\ &\Theta(w) = \Phi^+ \cap w(\Phi^-) && \theta(w) = \textstyle\sum\Theta(w) && \Theta_L(w) = \Phi^+ \cap w(\Phi^- \smallsetminus \Phi_L) && \theta_L(w) = \textstyle\sum\Theta_L(w) \end{aligned}$$

For any subset $\Psi \subset \Phi$, we define $\mathfrak{g}(\Psi) = \bigoplus_{\alpha\in\Psi}\mathfrak{g}_\alpha$. Next, let $B_w = wBw^{-1}$, and let U_w denote the unipotent radical of B_w. Its Lie algebra $\mathfrak{u}_w$ is described by $\mathfrak{u}_w = \mathfrak{g}(w(\Phi^+))$. Let $\langle\cdot,\cdot\rangle$ denote the Killing form. By rescaling if necessary, assume that $\langle 2\rho, \lambda\rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$.

Now, the category $\mathfrak{Coh}^B(X^\circ_w)$ is equivalent to the category $\mathfrak{Rep}(B_w \cap B)$ of representations of the isotropy group $B_w \cap B$. We define an s-structure on X°_w via this equivalence as follows:

$$\begin{aligned} \mathfrak{Coh}^B(X^\circ_w)_{\le n} &\simeq \{V \in \mathfrak{Rep}(B_w \cap B) \mid \langle\lambda, -2\rho\rangle \le n \text{ for all weights } \lambda \text{ occurring in } V\} \\ \mathfrak{Coh}^B(X^\circ_w)_{\ge n} &\simeq \{V \in \mathfrak{Rep}(B_w \cap B) \mid \langle\lambda, -2\rho\rangle \ge n \text{ for all weights } \lambda \text{ occurring in } V\} \end{aligned} \tag{1}$$

Lemma 3.1. *For any $v, w \in W$, there is a B_v-equivariant isomorphism $B_vwP/P \simeq \mathfrak{g}(v(\Theta_L(v^{-1}w)))$.*

Proof. We have $B_vwP/P = w \cdot B_{w^{-1}v}P/P \simeq w \cdot B_{w^{-1}v}/(B_{w^{-1}v} \cap P)$. Since $B_{w^{-1}v} \cap P$ contains the maximal torus T, the quotient $B_{w^{-1}v}/(B_{w^{-1}v} \cap P)$ can be identified with a quotient of $U_{w^{-1}v}$, and hence of $\mathfrak{u}_{w^{-1}v}$. Specifically, it is isomorphic to $\mathfrak{g}(w^{-1}v(\Phi^+) \smallsetminus \Phi_P) \simeq \mathfrak{g}(w^{-1}v(\Phi^+) \cap (\Phi^- \smallsetminus \Phi_L))$, so

$$B_vwP/P \simeq w \cdot \mathfrak{g}(w^{-1}v(\Phi^+) \cap (\Phi^- \smallsetminus \Phi_L)) \simeq \mathfrak{g}(v(\Theta_L(v^{-1}w))).$$ □

In the special case $v = ww_0$, where w_0 is the longest element of W, the set $v(\Theta_L(v^{-1}w))$ is given by

$$ww_0(\Theta_L(w_0)) = w(\Phi^-) \cap w(\Phi^- \smallsetminus \Phi_L) = w(\Phi^- \smallsetminus \Phi_L) = -\Pi_L(w) \sqcup \Theta_L(w).$$

Let $Y_w = B_{ww_0}wP/P$. Applying Lemma 3.1 with $v = 1$ and with $v = ww_0$, we obtain

$$X_w^\circ \simeq \mathfrak{g}(\Theta_L(w)) \qquad \text{and} \qquad Y_w \simeq X_w^\circ \oplus \mathfrak{g}(-\Pi_L(w)). \tag{2}$$

Finally, let $\mathcal{I}_w$ denote the ideal sheaf on G/P corresponding to X_w. Since Y_w is open, Proposition 2.1 tells us that $i_w^*\mathcal{I}_w|_{X_w^\circ} \simeq \bigoplus_{\alpha\in\Pi_L(w)} \mathcal{O}_{X_w^\circ}(\alpha)$. Since $\langle\alpha, -2\rho\rangle < 0$ for all $\alpha \in \Phi^+$, we see that $i_w^*\mathcal{I}_w|_{X_w^\circ} \in \mathfrak{Coh}^B(X_w^\circ)_{\le -1}$, and then Theorem 1.1 gives us an s-structure on G/P. Separately, Proposition 2.1 also tells us that $i_w^!\mathcal{O}_{G/P}[\operatorname{codim} X_w]$ is in $\mathfrak{Coh}^B(G/P)_{\le\langle\pi_L(w),2\rho\rangle} \cap \mathfrak{Coh}^B(G/P)_{\ge\langle\pi_L(w),2\rho\rangle}$. If w is the unique element of maximal length in its coset wW_L, then we have $\operatorname{codim} X_w = |\Phi^+| - \ell(w)$ and $\pi_L(w) = \pi(w)$. (See [3, Chap. 2].) Combining these observations gives us the following theorem.

Theorem 3.2. *There is a unique s-structure on G/P compatible with those on the various X_w°. If w is the unique element of maximal length in wW_L, then the staggered codimension of X_w, with respect to the dualizing complex $\mathcal{O}_{G/P}$, is given by* $\operatorname{scod} X_w = |\Phi^+| - \ell(w) + \langle\pi(w), 2\rho\rangle$. □

4. Main result

Theorem 4.1. *With respect to the s-structure and dualizing complex of Theorem 3.2, $\mathcal{D}^B(G/P)$ admits an artinian staggered t-structure. In particular, the set of simple staggered sheaves $\{\mathcal{IC}(X_w, \mathcal{O}_{X_w^\circ}(\lambda))\}$, where $\lambda \in \Lambda$, and w ranges over a set of coset representatives of W_L, forms a basis for $K^B(G/P)$.*

By the remarks in the introduction, this theorem follows from Proposition 4.6 below. Throughout this section, the notation "$u \cdot v$" for the product of $u, v \in W$ will be used to indicate that $\ell(uv) = \ell(u) + \ell(v)$. Note that if s is a simple reflection corresponding to a simple root α, $\ell(sw) > \ell(w)$ if and only if $\alpha \in \Pi(w)$.

Lemma 4.2. *Let s be a simple reflection, and let α be the corresponding simple root. If $\ell(sw) > \ell(w)$, then $\pi(sw) = s\pi(w) + \alpha$ and $\theta(sw) = s\theta(w) + \alpha$.*

Proof. Since $\Pi(s) = \Phi^+ \smallsetminus \{\alpha\}$, it is easy to see that if $\alpha \in \Pi(w)$, then $\Pi(sw) = s(\Pi(w) \smallsetminus \{\alpha\})$, and hence that $\pi(sw) = s(\pi(w) - \alpha) = s\pi(w) + \alpha$. The proof of the second formula is similar. □

Lemma 4.3. *For any $w \in W$, we have $\langle\pi(w), \theta(w)\rangle = 0$.*

Proof. We proceed by induction on $\ell(w)$. If $w = 1$, $\theta(w) = 0$, and the statement is trivial. If $\ell(w) \ge 1$, write $w = s\cdot v$ with s a simple reflection. Let α be the corresponding simple root. We have $\langle\pi(w), \theta(w)\rangle = \langle\pi(sv), \theta(sv)\rangle = \langle s\pi(v) + \alpha, s\theta(v) + \alpha\rangle$, and so

$$\langle\pi(w), \theta(w)\rangle = \langle s\pi(v), s\theta(v)\rangle + \langle s\pi(v), \alpha\rangle + \langle s\theta(v), \alpha\rangle + \langle\alpha, \alpha\rangle = \langle\pi(v), \theta(v)\rangle + \langle s(2\rho) + \alpha, \alpha\rangle.$$

Now, $\langle\pi(v), \theta(v)\rangle$ vanishes by assumption. Since s permutes $\Phi^+ \smallsetminus \{\alpha\}$, and $2\rho - \alpha$ is the sum of all roots in $\Phi^+ \smallsetminus \{\alpha\}$, we see that $s(2\rho - \alpha) = 2\rho - \alpha$. But $s(2\rho - \alpha) = s(2\rho) + \alpha$ as well, so we find that

$$\langle\pi(w), \theta(w)\rangle = \langle 2\rho - \alpha, \alpha\rangle = \langle s(2\rho - \alpha), \alpha\rangle = \langle 2\rho - \alpha, s\alpha\rangle = -\langle 2\rho - \alpha, \alpha\rangle.$$

Comparing the second and last terms above, we see that all these quantities vanish, as desired. □

Proposition 4.4. *If $\alpha \in \Pi(w)$ is a simple root, then $\langle\alpha, \theta(w)\rangle \le 0$.*

Proof. It is clear that it suffices to consider the case where W is irreducible. We proceed by induction on $\ell(w)$. When $w = 1$, $\theta(w) = 0$, so the statement holds trivially. Now, suppose $\ell(w) > 0$, and let t be a simple reflection such that $\ell(tw) < \ell(w)$. Let β be the simple root corresponding to t. We must consider four cases, depending on the form of tw.

Case 1. $w = t \cdot v$ with $\alpha \in \Pi(v)$. Then $\langle\alpha, \theta(tv)\rangle = \langle\alpha, t\theta(v) + \beta\rangle = \langle t\alpha, \theta(v)\rangle + \langle\alpha, \beta\rangle$, so $\langle\alpha, \theta(tv)\rangle = \langle\alpha - \langle\beta^\vee, \alpha\rangle\beta, \theta(v)\rangle + \langle\alpha, \beta\rangle = \langle\alpha, \theta(v)\rangle - \langle\beta^\vee, \alpha\rangle\langle\beta, \theta(v)\rangle + \langle\alpha, \beta\rangle$. We know that $\langle\beta^\vee, \alpha\rangle \le 0$ and $\langle\alpha, \beta\rangle \le 0$. The fact that $\ell(tv) > \ell(v)$ implies that $\beta \in \Pi(v)$, and $\alpha \in \Pi(v)$ by assumption, so $\langle\alpha, \theta(v)\rangle \le 0$ and $\langle\beta, \theta(v)\rangle \le 0$ by induction. The result follows.

In the remaining cases, we will have $\alpha \notin \Pi(tw)$. This implies that s and t do not commute. Let $N = \langle\alpha^\vee, \beta\rangle\langle\beta^\vee, \alpha\rangle$. We then have $N \in \{1, 2, 3\}$, with $N = 3$ occurring only in type G_2.

Case 2. $w = ts \cdot v$ with $\beta \in \Pi(v)$. We have $\langle\alpha, \theta(tsv)\rangle = \langle\alpha, t\theta(sv) + \beta\rangle = \langle\alpha, ts\theta(v) + t\alpha + \beta\rangle = \langle st\alpha, \theta(v)\rangle + \langle\alpha, t\alpha + \beta\rangle$. It is easy to check that $st\alpha = (N-1)\alpha - \langle\beta^\vee, \alpha\rangle\beta$, and hence that $\langle st\alpha, \theta(v)\rangle = (N-1)\langle\alpha, \theta(v)\rangle - \langle\beta^\vee, \alpha\rangle\langle\beta, \theta(v)\rangle$. Now, $\beta \in \Pi(v)$ by assumption, and $\alpha \in \Pi(v)$ since $\ell(sv) > \ell(v)$, so $\langle\alpha, \theta(v)\rangle \le 0$ and $\langle\beta, \theta(v)\rangle \le 0$ by induction. Clearly, $N - 1 \ge 0$ and $\langle\beta^\vee, \alpha\rangle < 0$, so $\langle st\alpha, \theta(v)\rangle \le 0$. Next, we have $t\alpha + \beta = \alpha - \langle\beta^\vee, \alpha\rangle\beta + \beta$, so $\langle\alpha, t\alpha + \beta\rangle = \langle\alpha, \alpha\rangle - \langle\beta^\vee, \alpha\rangle\langle\alpha, \beta\rangle + \langle\alpha, \beta\rangle = \frac{\langle\alpha,\alpha\rangle}{2}(2 - N + \langle\alpha^\vee, \beta\rangle)$.

Recall that $\langle \alpha^\vee, \beta\rangle \in \{-1, -N\}$, so $(2 - N + \langle \alpha^\vee, \beta\rangle)$ is either $1 - N$ or $2 - 2N$. In either case, we see that $\langle \alpha, t\alpha + \beta\rangle \le 0$. It follows that $\langle \alpha, \theta(w)\rangle \le 0$.

In the last two cases, we assume that $\beta \notin \Pi(stw)$. This implies that $w = tst \cdot v$ for some v. We also have $sw = stst \cdot v$, so it must be that $N \ge 2$.

Case 3. $w = tst \cdot v$ and $N = 2$. In this case, $sw = stst \cdot v = tsts \cdot v$, so $\ell(sv) > \ell(v)$, and hence $\alpha \in \Pi(v)$. Calculations similar to those above yield that $\theta(tstv) = tst\theta(v) + ts\beta + t\alpha + \beta$, and that $\langle \alpha, ts\beta + t\alpha + \beta\rangle = \langle \alpha, \beta\rangle - \frac{\langle \alpha,\alpha\rangle}{2}\langle \alpha^\vee, \beta\rangle = 0$. Thus, $\langle \alpha, \theta(tstv)\rangle = \langle \alpha, tst\theta(v)\rangle + \langle \alpha, ts\beta + t\alpha + \beta\rangle = \langle tst\alpha, \theta(v)\rangle$. Direct calculation shows that $tst\alpha = \alpha$ (regardless of whether α is a short root or a long root). Since $\alpha \in \Pi(v)$, $\langle \alpha, \theta(v)\rangle \le 0$ by induction, so $\langle \alpha, \theta(w)\rangle \le 0$ as well.

Case 4. $w = tst \cdot v$ and $N = 3$. Since we have assumed that W is irreducible, W must be of type G_2. Since $sw = stst\cdot v$, we must have $v \in \{1, s, st\}$, since $ststst$ is the longest word in W. First suppose $v = st$. Since sw is the longest word, we have $\Pi(w) = \{\alpha\}$, and hence $\theta(w) = 2\rho - \alpha$, so Lemma 4.2 implies that $\langle \alpha, \theta(w)\rangle = 0$. If $v = s$, direct calculation gives $\theta(w) = 2\rho - \alpha - s\beta$, and then that $\langle \alpha, \theta(w)\rangle = \langle \alpha, \beta\rangle < 0$. Finally, if $v = 1$, we find that $\theta(w) = 2\rho - \alpha - s\beta - st\alpha$, and again $\langle \alpha, \theta(w)\rangle < 0$. □

Proposition 4.5. *Let s be a simple reflection, corresponding to the simple root α. Let v, w be such that $\ell(vsw) = \ell(v) + 1 + \ell(w)$. Then* $\langle \pi(vw), 2\rho\rangle - \langle \pi(vsw), 2\rho\rangle = (1 - \langle \alpha^\vee, \theta(v^{-1})\rangle)\langle w^{-1}\alpha, 2\rho\rangle > 0$.

Proof. We proceed by induction on $\ell(v)$. First, suppose that $v = 1$. Note that $\theta(v^{-1}) = 0$. Since $2\rho = \pi(w) + \theta(w)$, Lemma 4.3 implies that $\langle \pi(w), 2\rho\rangle = \langle \pi(w), \pi(w)\rangle$. Similarly,

$$\begin{aligned}\langle \pi(sw), 2\rho\rangle &= \langle \pi(sw), \pi(sw)\rangle = \langle s\pi(w) + \alpha, s\pi(w) + \alpha\rangle \\ &= \langle s\pi(w), s\pi(w)\rangle + 2\langle s\pi(w), \alpha\rangle + \langle \alpha, \alpha\rangle = \langle \pi(w), \pi(w)\rangle + 2\langle \pi(w), s\alpha\rangle + \langle 2\rho, \alpha\rangle \\ &= \langle \pi(w), 2\rho\rangle - 2\langle \pi(w), \alpha\rangle + \langle \pi(w) + \theta(w), \alpha\rangle = \langle \pi(w), 2\rho\rangle - \langle \pi(w) - \theta(w), \alpha\rangle.\end{aligned}$$

It is easy to see that $\pi(w) - \theta(w) = w(2\rho)$, whence it follows that $\langle \pi(w), 2\rho\rangle - \langle \pi(sw), 2\rho\rangle = \langle w^{-1}\alpha, 2\rho\rangle$. Finally, the fact that $\ell(sw) > \ell(w)$ implies that $w^{-1}\alpha \in \Phi^+$, so $\langle w^{-1}\alpha, 2\rho\rangle > 0$.

Now, suppose $\ell(v) \ge 1$, and write $v = t \cdot x$, where t is a simple reflection with simple root β. Using the special case of the proposition that is already established, we find

$$\langle \pi(xsw), 2\rho\rangle - \langle \pi(txsw), 2\rho\rangle = \langle w^{-1}sx^{-1}\beta, 2\rho\rangle \qquad \text{and} \qquad \langle \pi(xw), 2\rho\rangle - \langle \pi(txw), 2\rho\rangle = \langle w^{-1}x^{-1}\beta, 2\rho\rangle.$$

Combining these with the fact that $sx^{-1}\beta = x^{-1}\beta - \langle \alpha^\vee, x^{-1}\beta\rangle\alpha$, we find

$$\begin{aligned}\langle \pi(txw), 2\rho\rangle - \langle \pi(txsw), 2\rho\rangle &= (\langle \pi(xw), 2\rho\rangle - \langle \pi(xsw), 2\rho\rangle) + (\langle w^{-1}sx^{-1}\beta, 2\rho\rangle - \langle w^{-1}x^{-1}\beta, 2\rho\rangle) \\ &= (1 - \langle \alpha^\vee, \theta(x^{-1})\rangle)\langle w^{-1}\alpha, 2\rho\rangle - \langle \alpha^\vee, x^{-1}\beta\rangle\langle w^{-1}\alpha, 2\rho\rangle = (1 - \langle \alpha^\vee, \theta(x^{-1}) + x^{-1}\beta\rangle)\langle w^{-1}\alpha, 2\rho\rangle.\end{aligned}$$

An argument similar to that of Lemma 4.2 shows that $\theta(x^{-1}) + x^{-1}\beta = \theta(x^{-1}t) = \theta(v^{-1})$, so the desired formula is established. Since $\ell(vs) > \ell(v)$, we also have $\ell(sv^{-1}) > \ell(v^{-1})$, and then Proposition 4.4 tells us that $\langle \alpha^\vee, \theta(v^{-1})\rangle \le 0$. Thus, $\langle \pi(vw), 2\rho\rangle - \langle \pi(vsw), 2\rho\rangle > 0$. □

The preceding proposition is a statement about a pair of adjacent elements with respect to the Bruhat order. It immediately implies that for any $v, w \in W$ with $v < w$ in the Bruhat order, $\langle \theta(v), 2\rho\rangle - \langle \theta(w), 2\rho\rangle > 0$. By Theorem 3.2, we deduce the following result, and thus establish Theorem 4.1.

Proposition 4.6. *If $X_v \subset \overline{X_w}$, then* $\operatorname{scod} X_v - \operatorname{scod} X_w \ge 2$. □

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