Cohomology of algebraic groups, Lie algebras, and related finite groups of Lie type
Part 1

Christopher P. Bendel

SE Lie Theory Workshop X
University of Georgia
June 10, 2018

Goals

Considering modular representation theory:

Fix an algebraically closed field k of characteristic $p > 0$

Goals:

- Talk about a variety of connections/relationships between cohomology/extensions for these various algebraic structures.
- Talk about a number of computational problems, particular ones where the answers remain incomplete.
- Demonstrate how those connections can be used toward making computations.

Outline

Part 1:

- Start at the beginning . . .
- Structures of interest: algebraic groups, Lie algebras, and finite groups of Lie type
- Modules
- Cohomology and Extensions
- Some basic tools
- Connections, connections, and more connections
- A few computations

Part 2:

- Computations and more computations

Caveats!

A few general caveats:

- Certainly not exhaustive!
- Apologies for missed results.
- A "coincidence": the problems I am most familiar with and find interesting, ...often happen to be the ones I have worked on

A couple technical caveats:

- There will be few comments on characteristic zero fields.
- Not always precise with statements of results.
- Terminology with multiple meanings (e.g., restricted and induction)

References

- *Representations of Algebraic Groups*, J. C. Jantzen, Mathematical Surveys and Monographs 107, AMS, 2003.
- *Modular Representations of Finite Groups of Lie Type*, J. E. Humphreys, LMS Lecture Note Series 326, Cambridge University Press, 2005.

G - an algebraic group scheme over k

- Functor from (commutative) k-algebras to groups
- Coordinate algebra: $k[G]$; commutative Hopf algebra
    - $G(A) = \mathrm{Hom}_{k-\mathrm{alg}}(k[G], A)$, for a k-algebra A
- Distribution algebra (or hyperalgebra): $\mathrm{Dist}(G) \subset k[G]^*$

Examples: Here A is a k-algebra

- The general linear group - GL_n:
  $GL_n(A) = \{n \times n \text{ invertible matrices with entries in } A\}$
- The $\mathbb{G}_a$: $\mathbb{G}_a(A) = (A, +)$
    - $k[\mathbb{G}_a] = k[t]$ - one variable polynomial ring
- $\mathbb{G}_m$: $\mathbb{G}_m(A) = (A^\times, *)$, $k[\mathbb{G}_m] = k[t, t^{-1}]$

Lie Algebras

$\mathfrak{g} = \text{Lie}(G)$ - The Lie algebra of G over k

- A p-restricted Lie algebra: $(-)^{[p]} : \mathfrak{g} \to \mathfrak{g}$

Examples:

- $\text{Lie}(GL_n) = \mathfrak{gl}_n = \{n \times n \text{ matrices over } k\}$
    - $[B, C] = BC - CB$
    - $B^{[p]} = B^p$ (matrix power)
- $\text{Lie}(\mathbb{G}_a) = k$ with trivial bracket and $[p]$-map

Enveloping Algebras

$U(\mathfrak{g})$ - the universal enveloping algebra of $\mathfrak{g}$

- In characteristic zero, $U(\mathfrak{g}) \cong \mathrm{Dist}(G)$

The p-restricted case:

$u(\mathfrak{g}) := U(\mathfrak{g})/(x^p - x^{[p]})$ - the restricted enveloping algebra

- Finite-dimensional, with dimension $p^{\dim(\mathfrak{g})}$
- There is an injection $u(\mathfrak{g}) \hookrightarrow \mathrm{Dist}(G)$

Frobenius Kernels

The Frobenius morphism of schemes: $F : G \to G$

$G_r := \ker F^r$ - scheme theoretic kernel

- G_r is a normal subgroup scheme of G
- $G_1 \subset G_2 \subset G_r \subset \cdots \subset G$
- $k[G_r]$ is finite-dimensional and local

Examples:

- $F : GL_n \to GL_n$ by $F((a_{ij})) = (a_{ij}^p)$
- $F : \mathbb{G}_a \to \mathbb{G}_a$ by $F(a) = a^p$
  - Comorphism $F^* : k[\mathbb{G}_a] \to k[\mathbb{G}_a]$ by $F(t) = t^p$
  - $k[\mathbb{G}_{a,r}] = k[t]/(t^{p^r})$

Connecting G_1 and $\mathfrak{g}$

Elementary Example:

- $k[\mathbb{G}_{a,1}] \cong k[t]/(t^p)$
- $u(\mathbb{G}_a) = U(\mathbb{G}_a)/(x^p - x^{[p]}) = k[x]/(x^p)$

General Fact:

$k[G_1]^* \cong u(\mathfrak{g})$ as Hopf algebras

Finite Groups

Recall Frobenius: $F : G \to G$

- $G(\mathbb{F}_p) := G^F$ or more generally
- $G(\mathbb{F}_q) := G^{F^r}$, $q = p^r$
  - $kG(\mathbb{F}_q)$ - the group algebra
- There exist "twisted" versions as well.

- $GL_n(\mathbb{F}_p)$ - $n \times n$ invertible matrices with entries in $\mathbb{F}_p$
- $\mathbb{G}_a(\mathbb{F}_p) = (\mathbb{F}_p, +)$
- $G_r(\mathbb{F}_p) = 1$

Representations - Algebraic Groups

General Assumption (any context): A module M will be a finite-dimensional vector space over k

Algebraic group G: a rational G-module M

- via an action of G on M
- via a group scheme homomorphism $G \to GL(M)$
- as a co-module for $k[G]$
- Gives rise to a $\text{Dist}(G)$-module

Representations - Lie algebras

Lie algebra $\mathfrak{g}$:

- Ordinary module: a $U(\mathfrak{g})$-module
    - or via an appropriate $\mathfrak{g}$-action
- Restricted module: a $u(\mathfrak{g})$-module
    - $\mathfrak{g}$-action that respects the $[p]$-mapping

Any G-module can be considered as a restricted $\mathfrak{g}$-module.

Note: Study of non-restricted representations for a p-restricted Lie algebra ...

Frobenius kernel G_r:

- As an algebraic group
- As a $k[G_r]^*$-module
- Any G-module is a G_r-module via restriction

G_1-modules are equivalent to restricted $\mathfrak{g}$-modules

Representations - Finite groups

Finite groups: $G(\mathbb{F}_q)$

- via an ordinary group homomorphism $G(\mathbb{F}_q) \to GL(M)$
- as a $kG(\mathbb{F}_q)$-module
  - the "defining characteristic" case
- Again, any G-module is a $G(\mathbb{F}_q)$-module via restriction

Note: Over characteristic zero, $kG(\mathbb{F}_q)$-modules are semisimple.

Can also consider the "non-defining characteristic" case: $kG(\mathbb{F}_q)$-modules where $q = p^r$ and the characteristic of k is prime but *not* p. Might or might not be semisimple . . .

New Modules from Old

M, N - (finite-dimensional) G-modules

- Direct Sums: $M \oplus N$
- Tensor Products: $M \otimes N$
- Dual Modules: $M^* := \mathrm{Hom}_k(M, k)$
- Frobenius Twists: $r \geq 1$, $F^r : G \to G$
    - $M^{(r)} := M$ with $g \in G$ acting via $F^r(g)$
    - If $N \cong M^{(r)}$, we may write $N^{(-r)}$ for M
- Induced Modules: $\mathrm{ind}_G^{G'}(M)$ for $G \subset G'$

Restrictions of twisted G-modules:

- Over G_r, $M^{(r)}$ is trivial - i.e., $k^{\oplus \dim M}$
- Over $G(\mathbb{F}_q)$, $M^{(r)} \cong M$

Cohomology

M - (finite-dimensional) G-module

$\mathrm{H}^i(G, M)$:

- ith right derived functor of the fixed point functor $(-)^G$ on M
- ith right derived functor of $\mathrm{Hom}_G(k, -)$ on M
    - $\mathrm{H}^i(G, M) \cong \mathrm{Ext}^i_G(k, M)$
- via the Hochschild complex $C^\bullet(G, M)$:
  $C^n(G, M) = M \otimes \bigotimes^n k[G]$

Cohomology - Ring Structure

$\mathrm{H}^{\bullet}(G,k) := \bigoplus_{i \geq 0} \mathrm{H}^i(G,k)$ has a ring structure via the cup product

- via the Hochschild complex or Yoneda splice (using extensions)
- graded commutative: for $a \in \mathrm{H}^i$ and $b \in H^j$, $ab = (-1)^{ij} ba$
  - commutative for $p = 2$
  - $\mathrm{H}^{2\bullet}(G,k)$ is commutative

Extensions

M, N - finite-dimensional G-modules

$\mathrm{Ext}^i_G(N, M)$:

- ith right derived functor of $\mathrm{Hom}_G(N, -)$ on M
- ith right derived functor of $\mathrm{Hom}_G(-, M)$ on N; if enough projectives exist
- Equivalence classes of extensions:

$$0 \to M \to C_1 \to \cdots \to C_i \to N \to 0$$

- $\mathrm{Ext}^i_G(N, M) \cong \mathrm{Ext}^i_G(k, N^* \otimes M) \cong \mathrm{H}^i(G, N^* \otimes M)$

Cohomology for G_r and $G(\mathbb{F}_q)$

Defined similarly

For G_r: can define via $k[G_r]^*$

For $G(\mathbb{F}_q)$:

- ordinary group cohomology
- can define via $kG(\mathbb{F}_q)$
- can use the bar resolution

Cohomology for Lie algebras

Warning: Need to be careful about context here.

Restricted cohomology: $\mathrm{H}^i(u(\mathfrak{g}), M) \cong \mathrm{H}^i(G_1, k)$

- Cohomology of the restricted enveloping algebra or, equivalently, the first Frobenius kernel

Ordinary cohomology: $\mathrm{H}^i(\mathfrak{g}, M) = \mathrm{H}^i(U(\mathfrak{g}), M)$

- Cohomology of the (full) enveloping algebra
- Can be computed using the cohomology of a complex $M \otimes \Lambda^\bullet(\mathfrak{g}^*)$
- $\mathrm{H}^i(\mathfrak{g}, M) = 0$ for $i > \dim \mathfrak{g}$

Cohomology Examples

For $\mathbb{G}_m$: (semi-simplicity)

- $\mathrm{H}^i(\mathbb{G}_m, M) = 0$ for $i > 0$ for any $\mathbb{G}_m$-module M

Cohomology Examples

For $\mathbb{G}_m$: (semi-simplicity)

- $\mathrm{H}^i(\mathbb{G}_m, M) = 0$ for $i > 0$ for any $\mathbb{G}_m$-module M

For $\mathbb{G}_a$:

- $p = 2$, $\mathrm{H}^\bullet(\mathbb{G}_a, k) = k[\lambda_1, \lambda_2, \ldots]$, with $\lambda_i \in \mathrm{H}^1$
- $p > 2$, $\mathrm{H}^\bullet(\mathbb{G}_a, k) = \Lambda^\bullet(\lambda_1, \lambda_2, \ldots) \otimes k[x_1, x_2, \ldots]$, with $\lambda_i \in \mathrm{H}^1$, $x_i \in \mathrm{H}^2$
- characteristic zero, $\mathrm{H}^\bullet(\mathbb{G}_a, k) = \Lambda^\bullet(k)$

- $p = 2$, $\mathrm{H}^\bullet(\mathbb{G}_{a,r}, k) = k[\lambda_1, \lambda_2, \ldots, \lambda_r]$, with $\lambda_i \in \mathrm{H}^1$
- $p > 2$, $\mathrm{H}^\bullet(\mathbb{G}_{a,r}, k) = \Lambda^\bullet(\lambda_1, \lambda_2, \ldots, \lambda_r) \otimes k[x_1, x_2, \ldots, x_r]$, with $\lambda_i \in \mathrm{H}^1$, $x_i \in \mathrm{H}^2$

Note: The cohomology of $\mathbb{G}_{a,r}$ is the same as that of the elementary abelian group $(\mathbb{Z}/p)^r$.

Reductive Groups

The radical of G: $R(G)$ - largest connected normal solvable subgroup of G

If the unipotent radical of $R(G)$ is trivial, we say G is reductive.

- e.g., GL_n

If $R(G)$ is trivial, we say G is semisimple.

- e.g., SL_n

For simplicity, generally assume G is semisimple with an irreducible root system and simply connected.

Classic Examples

Classic Matrix Examples:

- Type A_n: SL_{n+1}
- Type B_n: SO_{2n+1}
- Type C_n: Sp_{2n}
- Type D_n: SO_{2n}

and the exceptional groups in types E_6, E_7, E_8, F_4, and G_2.

Root Systems

- T: maximal torus (diagonalizable subgroup isomorphic to $\mathbb{G}_m^n$) of rank n in G
  - e.g., the diagonal matrices in SL_{n+1}
- Φ: irreducible root system associated to (G, T)
  - weights for the adjoint action of T on $\mathfrak{g}$
  - Φ^+, Φ^-: positive and negative roots, respectively
  - $S = \{\alpha_1, \ldots, \alpha_n\}$ - simple roots
- $B = T \ltimes U$: a Borel subgroup associated to the negative roots
  - U: product of negative root subgroups
  - e.g., the lower triangular matrices in SL_{n+1}
- W - the Weyl group generated by simple reflections

Root System Geometry

- $\mathbb{E}$: the Euclidean space spanned by Φ with inner product $\langle\ ,\ \rangle$
- The weight lattice: $X(T) = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$, where the fundamental dominant weights $\omega_i \in \mathbb{E}$ are defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, $1 \leq i, j \leq n$.
- coroots: $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$
- The dominant weights: $X(T)_+ := \{\lambda \in X(T) : \langle \lambda, \alpha^\vee \rangle \geq 0\ \forall\ \alpha \in S\} = \mathbb{N}\omega_1 \oplus \cdots \oplus \mathbb{N}\omega_n$
- The p^r-restricted weights: $X_r(T) := \{\lambda \in X(T) : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r\ \forall\ \alpha \in S\}$
- The Weyl weight: $\rho = \dfrac{1}{2} \displaystyle\sum_{\alpha \in \Phi^+} \alpha$
- The maximal short root: α_0

The Coxeter Number

$h := \langle \rho, \alpha_0^\vee \rangle + 1$

Φ	h
A_n	$n+1$
B_n	$2n$
C_n	$2n$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	12
G_2	6

h tends to separate "small" primes from "large" primes

Key G-Modules

For $\lambda \in X(T)_+$:

- Induced modules: $H^0(\lambda) := \text{ind}_B^G(\lambda)$
  - On the right, λ denotes the 1-dim module k_λ with U acting trivially and T by λ
- Weyl modules: $V(\lambda) := H^0(-w_0(\lambda))^*$
  - w_0 denotes the longest word in W
- Simple modules: $L(\lambda)$ - arise as
  - the socle of $H^0(\lambda)$
  - the head of $V(\lambda)$
- In characteristic zero, $H^0(\lambda)$ and $V(\lambda)$ are simple.

Steinberg's Tensor Product Theorem

Given $\lambda \in X(T)_+$, we may write it as

$$\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \cdots + p^m\lambda_m$$

where $\lambda_i \in X_1(T)$. Then

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes L(\lambda_2)^{(2)} \otimes \cdots \otimes L(\lambda_m)^{(m)}$$

Restricting to G_r or $G(\mathbb{F}_q)$

Question: What happens to a simple G-module $L(\lambda)$ upon restriction to G_r or $G(\mathbb{F}_q)$?

A first connection between G_r and $G(\mathbb{F}_q)$:

In either case, the set of irreducible modules is precisely

$$\{L(\lambda) : \lambda \in X_r(T)\}.$$

That is, the simples corresponding to the p^r-restricted weights.

Goal Revisited

Study cohomology and extensions over

- G, $\mathfrak{g}$, G_r, $G(\mathbb{F}_q)$
- B, $\mathfrak{b}$, B_r, $B(\mathbb{F}_q)$
- U, $\mathfrak{u}$, U_r, $U(\mathbb{F}_q)$

Main modules of interest:

- k, $L(\lambda)$, $H^0(\lambda)$, $V(\lambda)$ for $\lambda \in X(T)_+$

Underlying Theme

Identify relationships between cohomology groups for these various structures.

Apply computations from one realm to make computations in another.

A fundamental question: determine whether a cohomology group is non-zero.

M - a G-module

Recall the twisted module $M^{(r)}$

As r increases, the cohomology stabilizes. i.e., $\exists$ R, depending on i such that, for $r \geq R$,

$$\mathrm{H}^i(G, M^{(r)}) \cong \mathrm{H}^i(G, M^{(r+1)})$$

- Cline-Parshall-Scott-van der Kallen '77
- B-Nakano-Pillen '14

Connections: G and B

M, N - G-modules, $i \geq 0$

- $\operatorname{Ext}^i_G(M, N) \cong \operatorname{Ext}^i_B(M, N)$
- $\mathrm{H}^i(G, M) \cong \mathrm{H}^i(B, M)$
- Also holds for a *standard parabolic* between B and G
- Uses Kempf's vanishing theorem:

$$R^i \operatorname{ind}_B^G(\lambda) = 0 \text{ for } i > 0, \lambda \in X(T)_+$$

- Cline-Parshall-Scott-van der Kallen '77

M - B-module, $i \geq 0$

- $\mathrm{H}^i(B, M) \cong (\mathrm{H}^i(U, M))^T$

Recall: $B = T \ltimes U$

The power of spectral sequences ...

The Lyndon-Hochschild-Serre Spectral Sequence

Applied to $U \trianglelefteq B$:

$$E_2^{m,n} = H^m(B/U, H^n(U, M)) \Rightarrow H^{m+n}(B, M)$$

The abutment (right-hand side) is a "limit" of the left-hand side after a sequence of maps:

$d_2 : E^{m,n} \to E^{m+2,n-1}$
$d_3 : E^{m,n} \to E^{m+3,n-2}$ ...

The E_2-page

This is a first quadrant spectral sequence

$$\begin{array}{llll} \ldots & & & \\ H^0(B/U, H^2(U,M)) & H^1(B/U, H^2(U,M)) & H^2(B/U, H^2(U,M)) & \ldots \\ H^0(B/U, H^1(U,M)) & H^1(B/U, H^1(U,M)) & H^2(B/U, H^1(U,M)) & \ldots \\ H^0(B/U, H^0(U,M)) & H^1(B/U, H^0(U,M)) & H^2(B/U, H^0(U,M)) & \ldots \end{array}$$

The E_2-page

This is a first quadrant spectral sequence

$$\ldots$$

$$\begin{array}{llll}
H^0(B/U, H^2(U,M)) & H^1(B/U, H^2(U,M)) & H^2(B/U, H^2(U,M)) & \ldots \\
H^0(B/U, H^1(U,M)) & H^1(B/U, H^1(U,M)) & H^2(B/U, H^1(U,M)) & \ldots \\
H^0(B/U, H^0(U,M)) & H^1(B/U, H^0(U,M)) & H^2(B/U, H^0(U,M)) & \ldots
\end{array}$$

Since $B/U \cong T$ (a torus), $H^i(B/U, H^j(U,M)) = 0$ for all $i > 0$ and $j \geq 0$

So the sequence really looks like . . .

$\ldots$

$H^0(B/U, H^2(U, M))$ 0 0 0 0 $\ldots$

$H^0(B/U, H^1(U, M))$ 0 0 0 0 $\ldots$

$H^0(B/U, H^0(U, M))$ 0 0 0 0 $\ldots$

And so all the differentials vanish giving

$$\begin{aligned} H^n(B, M) = H^{0+n}(B, M) &= H^0(B/U, H^n(U, M)) \\ &= H^0(T, H^n(U, M)) = H^n(U, M)^T \end{aligned}$$

Connections: Algebraic Groups and Frobenius Kernels

Recall: there is a chain of subgroups

$$G_1 \subset G_2 \subset \cdots \subset G$$

For any G-module M, the inclusion $G_r \hookrightarrow G$ induces a map in cohomology: $\mathrm{H}^i(G, M) \to \mathrm{H}^i(G_r, M)$

Connections: Algebraic Groups and Frobenius Kernels

Recall: there is a chain of subgroups

$$G_1 \subset G_2 \subset \cdots \subset G$$

For any G-module M, the inclusion $G_r \hookrightarrow G$ induces a map in cohomology: $\mathrm{H}^i(G, M) \to \mathrm{H}^i(G_r, M)$

For all $i \geq 0$,

- $\mathrm{H}^i(G, M) \cong \varprojlim \mathrm{H}^i(G_r, M)$, any G-module M
- $\mathrm{H}^i(B, M) \cong \varprojlim \mathrm{H}^i(B_r, M)$, any B-module M
- Again, holds for parabolics more generally.
- Cline-Parshall-Scott '80, Friedlander-Parshall '87, van der Kallen (per Jantzen '03)

Connections: Algebraic Groups and Frobenius Kernels/Finite Groups

By Generalized Frobenius Reciprocity,

- $\mathrm{H}^i(G_r, M) \cong \mathrm{H}^i(G, \mathrm{ind}_{G_r}^G(M))$, for a G_r-module M
- $\mathrm{H}^i(G(\mathbb{F}_q), M) \cong \mathrm{H}^i(G, \mathrm{ind}_{G(\mathbb{F}_q)}^G(M))$, for a $G(\mathbb{F}_q)$-module M

But these G-modules are infinite-dimensional.

Yet still useful. More on that coming . . .

Connections: Generic Cohomology

For a G-module M, the embedding $G(\mathbb{F}_q) \hookrightarrow G$ induces a restriction map in cohomology:

$$\mathrm{H}^i(G, M) \to \mathrm{H}^i(G(\mathbb{F}_q), M)$$

If we twist the module M and allow r in $q = p^r$ to grow, the map is an isomorphism.

Connections: Generic Cohomology

For a G-module M, the embedding $G(\mathbb{F}_q) \hookrightarrow G$ induces a restriction map in cohomology:

$$\mathrm{H}^i(G, M) \to \mathrm{H}^i(G(\mathbb{F}_q), M)$$

If we twist the module M and allow r in $q = p^r$ to grow, the map is an isomorphism.

For sufficiently large r and s (depending on i and $\ldots$),

$$\mathrm{H}^i(G, M^{(s)}) \cong \mathrm{H}^i(G(\mathbb{F}_q), M^{(s)}) \cong \mathrm{H}^i(G(\mathbb{F}_q), M).$$

i.e., As r increases, $\mathrm{H}^i(G(\mathbb{F}_q), M)$ stabilizes to a "generic" value

- Cline-Parshall-Scott-van der Kallen '77
- B-Nakano-Pillen '14
- Questions/Problems:
  - Identifying sharp bounds for stabilization
  - Computing generic cohomology

Direct G to $G(\mathbb{F}_q)$ Connections

Andersen '84: $\lambda, \mu \in X_r(T)$ with $\langle \lambda + \mu, \alpha_0^\vee \rangle < p^r - p^{r-1} - 1$,

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu))$$

I.e., $\lambda + \mu$ is not too large relative to p

Taking $\lambda = 0$, gives a condition for

$$\mathrm{H}^1(G, L(\mu)) \cong \mathrm{H}^1(G(\mathbb{F}_q), L(\mu))$$

Induction and Truncation

Recall: $\mathrm{Ext}^i_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^i_G(L(\lambda), L(\mu) \otimes \mathrm{ind}_{G(\mathbb{F}_q)}^G(k))$

B-Nakano-Pillen '01, '02, '04: Truncate $\mathrm{ind}_{G(\mathbb{F}_q)}^G(k)$ by taking the largest submodule whose composition factors have highest weights that are "small" ...

For $p \geq 3(h-1)$,

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \bigoplus_{\nu \in \Gamma} \mathrm{Ext}^1_G(L(\lambda) \otimes L(\nu)^{(r)}, L(\mu) \otimes L(\nu)).$$

$\Gamma := \{\nu \in X(T)_+ : \langle \nu, \alpha_0^\vee \rangle < h\}$

More G to $G(\mathbb{F}_q)$-connections

If the right-hand Ext-groups vanish for $\nu \neq 0$, then

$$\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu))$$

For example: $\lambda = \lambda_0 + p\lambda_1 + \cdots p^{r-1}\lambda_{r-1}$,
$\mu = \mu_0 + p\mu_1 + \cdots + p^{r-1}\mu_{r-1}$

- Holds if $\lambda_{r-1} = \mu_{r-1}$
- E.g, if $\lambda = \mu$
- Fix λ and μ, can let r increase until true

Question: What about small primes?

Can get some of this assuming p^r is sufficiently large, with p "small" (e.g., B-Nakano-Pillen '06).

Shifting Weights

Even if $\operatorname{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \ncong \operatorname{Ext}^1_G(L(\lambda), L(\mu))$, we can "shift" weights . . .

B-Nakano-Pillen '02, '06: Still $p \geq 3(h-1)$ (or p^r large) .

For $r \geq 3$, given $\lambda, \mu \in X_r(T)$, there exist $\tilde{\lambda}, \tilde{\mu} \in X_r(T)$ such that

$$\operatorname{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \operatorname{Ext}^1_G(L(\tilde{\lambda}), L(\tilde{\mu}))$$

Corollary:

$$\begin{aligned} &\max\{\dim_k \operatorname{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \mid \lambda, \mu \in X_r(T)\} \\ &\quad = \max\{\dim_k \operatorname{Ext}^1_G(L(\lambda), L(\mu)) \mid \lambda, \mu \in X_r(T)\}. \end{aligned}$$

Generic Cohomology Returns

Parshall-Scott-Stewart '13: For sufficiently large r (depending on m and the root system), given $\lambda \in X_r(T)$, there exists $\tilde{\lambda} \in X_r(T)$ such that

$$\mathrm{H}^m(G(\mathbb{F}_q), L(\lambda)) \cong \mathrm{H}^m(G, L(\tilde{\lambda}))$$

i.e., can get generic cohomology *without* the twisting on the right.

Note: there is no condition on the prime here.

Connecting G and G_r

Can use LHS for $G_r \trianglelefteq G$ ...

Connecting G and G_r

Can use LHS for $G_r \trianglelefteq G$ ...

Using $\mathrm{ind}_{G_r}^{G}$ (and Andersen '84), one can get (B-Nakano-Pillen '02) a similar sort of result for $p \geq 3(h-1)$:

$$\mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G}(L(\lambda), L(\mu)) \oplus (\text{Remainder Term})$$

Example: for $r \geq 2$
$\lambda = \lambda_0 + p\lambda_1 + \cdots p^{r-1}\lambda_{r-1} = \dot{\lambda} + p^{r-1}\lambda_{r-1}$,
$\mu = \mu_0 + p\mu_1 + \cdots p^{r-1}\mu_{r-1} = \dot{\mu} + p^{r-1}\mu_{r-1}$

If $\dot{\lambda} \neq \dot{\mu}$, then

$$\mathrm{Ext}^1_{G_r}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_{G}(L(\lambda), L(\mu))$$

Connections: Frobenius Kernels and Lie Algebras

Let G be arbitrary for this slide and M - G_r-module.

The May spectral sequence leads to spectral sequences:

- $E_1^{\bullet,\bullet} = S(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)^{(1)} \otimes \cdots \otimes S(\mathfrak{g}^*)^{(r-1)} \otimes M \Rightarrow \mathrm{H}^\bullet(G_r, M)$, $p = 2$
- $E_1^{\bullet,\bullet} = \Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)^{(1)} \otimes \cdots \otimes \Lambda(\mathfrak{g}^*)^{(r-1)} \otimes S(\mathfrak{g}^*)^{(1)} \otimes S(\mathfrak{g}^*)^{(2)} \otimes \cdots \otimes S(\mathfrak{g}^*)^{(r)} \otimes M \Rightarrow \mathrm{H}^\bullet(G_r, M)$, $p > 2$.
- $\exists$ G-versions also

The $p > 2$ case may be refined to (Friedlander-Parshall '86):

$$E_2^{2i,j} = S^i(\mathfrak{g}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{g}, M) \Rightarrow \mathrm{H}^{2i+j}(G_1, M)$$

Until Tomorrow

To be continued . . .