Cohomology of algebraic groups, Lie algebras, and related finite groups of Lie type
Part 2

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SE Lie Theory Workshop X
University of Georgia
June 11, 2018

Context From Yesterday

Generally thinking of G as an almost simple, semisimple, simply connected algebraic group (scheme) over an algebraically closed field of characteristic $p > 0$.

Classic Matrix Examples:

- Type A_n: SL_{n+1}
- Type B_n: SO_{2n+1}
- Type C_n: Sp_{2n}
- Type D_n: SO_{2n}

and the exceptional groups in types E_6, E_7, E_8, F_4, and G_2.

Notation

- Φ : the irreducible root system
- Φ^+: positive roots
- S: simple roots
- $X(T)$: weight lattice
- $X(T)_+$: dominant weights
- $X_r(T)$: p^r-restricted weights
- T: maximal torus
- $B = T \ltimes U$: a Borel subgroup associated to the negative roots
- W: the Weyl group
- The Weyl weight: $\rho = \dfrac{1}{2} \displaystyle\sum_{\alpha \in \Phi^+} \alpha$
- The maximal short root: α_0

The Coxeter Number

$h := \langle \rho, \alpha_0^\vee \rangle + 1$

Φ	h
A_n	$n+1$
B_n	$2n$
C_n	$2n$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	12
G_2	6

h tends to separate "small" primes from "large" primes

Goal Revisited

Study cohomology and extensions over

- G, $\mathfrak{g}$, G_r, $G(\mathbb{F}_q)$
- B, $\mathfrak{b}$, B_r, $B(\mathbb{F}_q)$
- U, $\mathfrak{u}$, U_r, $U(\mathbb{F}_q)$

Here $q = p^r$

Main modules of interest:

- k, $L(\lambda)$, $H^0(\lambda)$, $V(\lambda)$ for $\lambda \in X(T)_+$

Yesterday

- Structures of interest:
  - Algebraic groups
  - Frobenius kernels
  - Associated Lie algebras
  - Associated finite groups of Lie type
- Modules
- Cohomology and Extensions
- Some basic tools
- Connections in cohomology

Some of Yesterday's Connections

- $\mathrm{H}^i(G, M) \cong \mathrm{H}^i(B, M)$ for a G-module M
- $\mathrm{H}^i(B, M) \cong \mathrm{H}^i(U, M)^T$ for a B-module M
- $\mathrm{H}^i(B, M) \cong \varprojlim \mathrm{H}^i(B_r, M)$ for a B-module M
- $\mathrm{H}^i(G(\mathbb{F}_q), k) \cong \mathrm{H}^i(G, \mathrm{ind}_{G(\mathbb{F}_q)}^G(k))$
- Lyndon-Hochshild-Serre spectral sequence

Today

Goal: See some of these connections at work . . . and identify what we know and don't know . . .

Outline:

- Cohomology with trivial coefficients
- Cohomology for some standard non-trivial modules
- Extensions with non-trivial modules - particularly self-extensions
- Bounding dimensions of cohomolgy/extension groups

Computations: Trivial Coefficients

Start by considering cohomology with coefficients in k.

E.g.,

- $\mathrm{H}^i(G, k)$, $\mathrm{H}^i(B, k)$, $\ldots$
- $\mathrm{H}^i(G_r, k)$, $\ldots$
- $\mathrm{H}^i(\mathfrak{g}, k)$, $\ldots$
- $\mathrm{H}^i(G(\mathbb{F}_q), k)$, $\ldots$

In addition to individual computations, would potentially like to understand the ring structure and related properties.

G- and B-Cohomology

Using root combinatorics and the aforementioned relationship between G- and B-cohomology,

$$\mathrm{H}^i(G,k) \cong \mathrm{H}^i(B,k) = \begin{cases} k & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

- Cline-Parshall-Scott-van der Kallen '77
- Holds for parabolics in between

Compare: U-Cohomology

$\mathrm{H}^\bullet(U, k)$ is certainly more complex.

E.g., For SL_2, $U = \mathbb{G}_a$,

Recall ($p > 2$):

$$\mathrm{H}^\bullet(U, k) = \Lambda^\bullet(\lambda_1, \lambda_2, \dots) \otimes k[x_1, x_2, \dots],$$

with $\lambda_i \in \mathrm{H}^1$, $x_i \in \mathrm{H}^2$

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Recent preprint of Friedlander:

$$\mathrm{H}^i(U,k) \cong \varprojlim \mathrm{H}^i(U_r, k)$$

Some progress towards the SL_3 case.

Very much an open problem.

First Frobenius kernels

We have good information for primes that are not too small relative to the root system.

For $p > h$, Andersen-Jantzen '84 and Friedlander-Parshall '86

- $\mathrm{H}^\bullet(B_1, k) \cong \mathrm{H}^\bullet(U_1, k)^{T_1} \cong S^\bullet(\mathfrak{u}^*)^{(1)}$
  - generators in degree 2
  - $E_2^{2i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{u}, k) \Rightarrow \mathrm{H}^{2i+j}(U_1, k)$
- $\mathrm{H}^i(G_1, k)^{(-1)} \cong \mathrm{ind}_B^G \left(S^{\frac{i}{2}}(\mathfrak{u}^*) \right)$
- $\mathrm{H}^\bullet(G_1, k) \cong k[\mathcal{N}]$ as algebras
  - $\mathcal{N} \subset \mathfrak{g}$ is the set of nilpotent elements in $\mathfrak{g}$

Small Primes?

Some small prime computations of Andersen-Jantzen '84

B-Nakano-Parshall-Pillen '14 considered $\mathrm{H}^\bullet(G_1, k)$ for small primes

Idea: Replace B with a parabolic $P_J \subset G$ corresponding to a certain set J of simple roots. Need some information:

- $R^i \operatorname{ind}_{P_J}^G(S^\bullet(\mathfrak{u}_J^*)) = 0$ for $i > 0$
- $G \cdot \mathfrak{u}_J$ is a normal subvariety of $\mathcal{N}$

If these hold (not known in general), ...

G_1-Cohomology for Small Primes

We would conclude that (for “very good” primes)

- $\mathrm{H}^{2\bullet+1}(G_1, k) = 0$
- $\mathrm{H}^{2\bullet}(G_1, k) \cong \mathrm{ind}_{P_j}^{G} S^{\bullet}(\mathfrak{u}_J^*) \cong k[G \cdot \mathfrak{u}_J]$

- Know for roughly $p > h/2$.
- But still open for smaller!
- Alternate answer in *very bad* case for SL_n with p dividing n.
- And what about $\mathrm{H}^{\bullet}(B_1, k)$ for small primes?

Higher r

For higher r, much less is known

- Degree ≤ 3 computations in a more general context - more later
- Andersen-Jantzen '84 - some SL_2-computations
- Kaneda-Shimada-Tezuka-Yagita '90 - $\mathrm{H}^i(B_r, k)$ for $p > h$
  - SL_n: $i \leq 2p - 1$
  - SL_3: all i for $r = 2$
- Ngo '13 - $G = SL_2$
  - U_r, B_r, G_r in all degrees
  - In fact, for some non-trivial modules
  - Ring structure of $\mathrm{H}^\bullet(U_r, k)$, but only of the reduced rings for B_r and G_r - which are shown to be Cohen-Macauley
- Friedlander - U_r for $G = SL_3$
  - Almost a computation of $\mathrm{H}^\bullet(U_r, k)$ modulo nilpotents

Higher r continued

Recent preprint of Ngo: in small degrees

For $i < \frac{p}{c}$, c depending on the root system (from 1 to 6)

$$\mathrm{H}^i(B_r, k) \cong \mathrm{H}^i(B_1, k)^{(r-1)} \cong \begin{cases} S^{\frac{i}{2}}(\mathfrak{u}^*)^{(r)} & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

$$\mathrm{H}^i(G_r, k)^{(-r)} \cong \mathrm{ind}_B^G \left(\mathrm{H}^i(B_r, k)^{(-r)} \right)$$

In general?

The Induction Question

When is

$$\mathrm{H}^i(G_r, k)^{(-r)} \cong \mathrm{ind}_B^G \left(\mathrm{H}^i(B_r, k)^{(-r)}\right)?$$

More generally, for a B-module M (e.g., simple), is

$$\mathrm{H}^i(G_r, \mathrm{ind}_B^G(M))^{(-r)} \cong \mathrm{ind}_B^G \left(\mathrm{H}^i(B_r, M)^{(-r)}\right)?$$

The induction spectral sequence:

$$E_2^{i,j} = R^i \,\mathrm{ind}_B^G \left(\mathrm{H}^i(B_r, M)^{(-r)}\right) \Rightarrow \mathrm{H}^i(G_r, \mathrm{ind}_B^G(M))^{(-r)}$$

Idea: Try to show this collapses.

Using Geometry for G_r

Cohomological Support Varieties: let G be a general algebraic group here

- Friedlander-Suslin '96 - $\mathrm{H}^\bullet(G_r, k)$ is finitely generated

$V_r(G) := \mathrm{Spec}\left(\mathrm{H}^{2\bullet}(G_r, k)\right)$

- Suslin-Friedlander-B '97 - For classical G (including B and U)
  - $V_r(G) \cong C_r(\mathcal{N}_p(\mathfrak{g}))$
  - r-tuples of pairwise commuting, p-nilpotent elements
  - Quite general for p good - McNinch '02, Sobaje '13

Support varieties for modules ... and other contexts ...

Lie Algebra Cohomology

$\mathrm{H}^\bullet(\mathfrak{g}, k) =?$

Small degree generalities:

- $\mathrm{H}^0(\mathfrak{g}, k) = k$
- $\mathrm{H}^1(\mathfrak{g}, k) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$
- $\mathrm{H}^2(\mathfrak{g}, k)$ may be identified with central extensions of $\mathfrak{g}$ by k
- $\mathrm{H}^i(\mathfrak{g}, k) = 0$ for $i > \dim \mathfrak{g}$

Not many explicit computations for $\mathfrak{g}$ in this context.

But there are for $\mathfrak{u}$ . . .

Kostant's Theorem

For $p \geq h-1$ (or characteristic zero), as a T-module

$$\mathrm{H}^i(\mathfrak{u}, k) \cong \bigoplus_{\ell(w)=i} -w \cdot 0$$

- ℓ - length function on W
- Dot action: $w \cdot \lambda = w(\lambda + \rho) - \rho$
- Friedlander-Parshall '86, Polo-Tilouine '02, UGA Vigre Algebra Group '09
- More general versions for $\mathfrak{u}_J$ associated to a parabolic and for simple G-modules

Small Primes

For $p < h - 1$, the UGA VIGRE Algebra Group showed that there is always more cohomology in a global sense:

$$\operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}, k) \neq \operatorname{ch} \mathrm{H}^\bullet(\mathfrak{u}, \mathbb{C})$$

- One can construct examples of "extra" cohomology classes in some degrees
- There is not necessarily extra cohomology in every degree
- For $\mathfrak{u}_J$, it remains an open question to understand when extra cohomology will arise.

$\mathrm{H}^i(\mathfrak{u}, k)$ for Small Primes Continued

$\mathrm{H}^i(\mathfrak{u}, k) = ?$

- There are computer calculations for small rank cases
- Fully known for degree 0, 1, 2 - Jantzen '91, B-Nakano-Pillen '07, Wright '11
- Degree 3 - known for $p \geq 5$ (and sometimes 7) - B-Nakano-Pillen '16
- Higher degrees???

Finite Groups

$\mathrm{H}^i(G(\mathbb{F}_q), k)$, $\mathrm{H}^i(B(\mathbb{F}_q), k)$, $\mathrm{H}^i(U(\mathbb{F}_q), k)$

Fits into the general context of finite groups

- Low degree interpretations
- There are many cohomology computations, particularly for "small" groups
  - e.g., Carlson '83 - $\mathrm{H}^i(SL_2(\mathbb{F}_q), k)$
- Cohomological support varieties ...

Finite Groups Continued

Another relationship:

$U(\mathbb{F}_q)$ is a p-Sylow subgroup of $G(\mathbb{F}_q)$, so

$$\mathrm{H}^i(G(\mathbb{F}_q), k) \hookrightarrow \mathrm{H}^i(B(\mathbb{F}_q), k) \hookrightarrow \mathrm{H}^i(U(\mathbb{F}_q), k)$$

Recall: Earlier observation on generic cohomology

- For sufficiently large r (depending on i), $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$
- When does this happen?

The Vanishing Range Problem

Dating back to Quillen '72, it's been observed that $\mathrm{H}^i(G(\mathbb{F}_q), k)$ is zero in "low" degrees. Specifically,

$$\mathrm{H}^i(G(\mathbb{F}_q), k) = 0 \text{ for } 0 < i < m$$

for some m depending on r.

Goal: Find a sharp bound m.

- an m where the above holds, but $\mathrm{H}^m(G(\mathbb{F}_q), k) \neq 0$
- i.e., the least positive degree with cohomology

Initial work involved finding larger and larger vanishing ranges ...

Vanishing Range Progress

- Quillen '72: $\mathrm{H}^i(GL_n(\mathbb{F}_q), k) = 0$ for $0 < i < r(p-1)$.
  - Observed the general phenomenon without specifics, nor sharpness
- Friedlander '75: Vanishing ranges for special orthogonal and symplectic groups
- Hiller '80: Vanishing ranges for G simply connected for all types.
- Friedlander-Parshall '83: $p > 2$, $\mathrm{H}^i(GL_n(\mathbb{F}_q), k) = \mathrm{H}^i(B(\mathbb{F}_q), k) = 0$ for $0 < i < r(2p-3)$
  - $\mathrm{H}^{r(2p-3)}(B(\mathbb{F}_q), k) \neq 0$
- Barbu '04: $\mathrm{H}^{r(2p-2)}(GL_n(\mathbb{F}_q), k) \neq 0$
- B-Nakano-Pillen '11, '12: For $p > h$ ...

Vanishing for $\mathrm{H}^i(G(\mathbb{F}_p), k)$

Simply connected case: $p > h$

Root System	p	Sharp Vanishing Bound D
$A_n, n \geq 4$	$p > n+2$	$2p-3$
$A_n, n \geq 3$	$p = n+2$	$p-2$
A_3	$p > 5$	$2p-6$
A_2	$p = 3k+1,\ k \geq 2$	$2p-6$
A_2	$p = 3k+2,\ k \geq 1$	$2p-3$
$B_n, n \geq 7$	$p > 2n$	$2p-3$
$B_n, n \in \{5,6\}$	$p > 13$	$2p-3$
$B_n, n \in \{5,6\}$	$p = 13$	$2p-5$
B_5	$p = 11$	$2p-7=3$
$B_n,\ n \in \{3,4\}$	$p > 2n$	$2p-8$
$C_n, n \geq 1$	$p > 2n$	$p-2$
$D_n, n \geq 4$	$p > 2n-2$	$2p-2n$

Vanishing for $\mathrm{H}^i(G(\mathbb{F}_p), k)$ Continued

Root System	p	Sharp Vanishing Bound D
E_6	$p > 13$	$2p - 3$
E_6	$p = 13$	$2p - 10 = 16$
E_7	$p > 23$	$2p - 3$
E_7	$p = 23$	$2p - 7 = 39$
E_7	$p = 19$	$2p - 9 = 27$
E_8	$p \geq 31$	$2p - 3$
F_4	$p \geq 13$	$2p - 9 \leq D \leq 2p - 3$
G_2	$p \geq 7$	$2p - 8 \leq D \leq 2p - 3$

Vanishing for $\mathrm{H}^i(G(\mathbb{F}_q), k)$ - i.e., $r > 1$

Still $p > h$

Insert an r in the above bounds.

- E.g., generic bound of $r(2p - 3)$
- Also not sharp for type B_n or E_6

Adjoint, simply laced cases (A_n, D_n, E_n):

- $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < r(2p - 3)$ and sharp

Vanishing Ranges Continued

- B-Nakano-Pillen '14:
  - $p = 2$: $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < r$
  - $p > 2$: $\mathrm{H}^i(G(\mathbb{F}_q), k) = 0$ for $0 < i < r(p-2)$
- Sprehn '15: Construction of non-trivial cohomology classes for $p > h$
  - $\mathrm{H}^{r(2p-3)}(G(\mathbb{F}_q), k) \neq 0$
  - $\mathrm{H}^{r(p-2)}(Sp_{2n}(\mathbb{F}_q), k) \neq 0$, Type C

Open issues:

- Small primes
- Some types in large primes

The BNP Proof Strategy

This is an application of the connections between groups:

- $\mathrm{H}^i(G(\mathbb{F}_q), k) \simeq \mathrm{H}^i(G, \mathrm{ind}_{G(\mathbb{F}_q)}^G(k))$
- Filter $\mathrm{ind}_{G(\mathbb{F}_q)}^G(k)$ by $H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}$ and show vanishing of

$$\mathrm{H}^i(G, H^0(\lambda) \otimes H^0(\lambda^*)^{(r)}) \cong \mathrm{Ext}^i_G(V(\lambda)^{(r)}, H^0(\lambda))$$

- For $r = 1$, use LHS spectral sequence for $G_1 \trianglelefteq G$:

$$E_2^{i,j} = \mathrm{Ext}^i_{G/G_1}(V(\lambda)^{(1)}, \mathrm{H}^j(G_1, H^0(\lambda))) \Rightarrow \mathrm{Ext}^{i+j}_G(V(\lambda)^{(1)}, H^0(\lambda))$$

- Investigate $\mathrm{H}^i(G_1, H^0(\lambda))$
  - More connections - see below - here is where $p > h$ comes in
- Use inductive arguments to get vanishing for higher r.

Non-trivial Modules

The Dream: Compute $\mathrm{Ext}^i_G(L(\lambda), L(\mu))$ for all $\lambda, \mu \in X(T)^+$

- Even for $i = 1$
- Related to the Lusztig Conjecture
- The Linkage Principle - Andersen '80
  - To be non-zero, λ and μ must be linked under the action of the affine Weyl group
- Special case: $\mathrm{H}^i(G, L(\lambda))$

Relating to G_1

Can try to use the LHS spectral sequence for $G_1 \trianglelefteq G$:

$$E_2^{i,j} = \mathrm{H}^i(G/G_1, \mathrm{H}^j(G_1, L(\lambda)) \Rightarrow \mathrm{H}^{i+j}(G, L(\lambda))$$

and the 5-term exact sequence for low degrees:

$$\begin{aligned} 0 &\to \mathrm{H}^i(G/G_1, L(\lambda)^{G_1}) \to \mathrm{H}^1(G, L(\lambda)) \to \left(\mathrm{H}^1(G_1, L(\lambda))^{G/G_1}\right) \\ &\to \mathrm{H}^2(G/G_1, L(\lambda)^{G_1}) \to \mathrm{H}^2(G, L(\lambda)) \end{aligned}$$

E.g., if $L(\lambda)^{G_1} = 0$, then $\mathrm{H}^1(G, L(\lambda)) \cong (\mathrm{H}^1(G_1, L(\lambda))^{G/G_1})$

Similar idea for $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$

See for example Donkin '82, Andersen '84, Jantzen '91

Relating to Other Modules

Consider the SES (for $\lambda \neq 0$)

$$0 \to L(\lambda) \to H^0(\lambda) \to H^0(\lambda)/L(\lambda) \to 0$$

and the associated LES (or more generally for Ext)

$$\begin{aligned} 0 &\to L(\lambda)^G \to H^0(\lambda)^G \to (H^0(\lambda)/L(\lambda))^G \\ &\to \mathrm{H}^1(G, L(\lambda)) \to \mathrm{H}^1(G, H^0(\lambda)) \to \mathrm{H}^1(G, H^0(\lambda)/L(\lambda)) \cdots \end{aligned}$$

For degree 1, $\mathrm{H}^1(G, H^0(\lambda)) = 0$, and so

$$\mathrm{H}^1(G, L(\lambda)) \cong (H^0(\lambda)/L(\lambda))^G$$

More generally, for $\mu \not> \lambda$

$$\mathrm{Ext}^1_G(L(\mu), L(\lambda)) \cong \mathrm{Hom}_G(L(\mu), H^0(\lambda)/L(\lambda))$$

and a higher degree version ...

Good Filtrations

The aforementioned vanishing fact is a special case of

$$\mathrm{Ext}^i_G(V(\lambda), H^0(\mu)) = \begin{cases} k & \text{if } i = 0 \text{ and } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

That follows by relating G-cohomology to B-cohomology . . .

A G-module M is said to have a *good filtration* if it has a a filtration by $H^0(\lambda)$s. That holds iff

$$\mathrm{Ext}^i_G(V(\lambda), M) = 0$$

for all $i > 0$ and $\lambda \in X(T)_+$.

B-Cohomology

For $\lambda \in X(T)$, $\mathrm{H}^i(B, \lambda) =?$

Characteristic zero: non-zero only for $\lambda = w \cdot 0$, $w \in W$, and in a single degree:

$$\mathrm{H}^{\ell(w)}(B, \lambda) = k$$

Characteristic p: there is more, but only known in small degrees

Degree 1: Andersen '84

$$\mathrm{H}^1(B, -p^m\alpha) = k$$

for $m \geq 0$, $\alpha \in S$

B-Cohomology Continued

- Degree 2: Andersen '84, O'Halloran '83, B-Nakano-Pillen '07, Wright '11
  - One-dimensional for (with $\alpha, \beta \in S$)

$$\lambda = \begin{cases} p^i w \cdot 0, i \geq 0, \ell(w) = 2 \\ -p^i \alpha, i > 0 \\ -p^i \alpha - p^j \beta, 0 \leq i < j \end{cases}$$

- Degree 3: Andersen-Rian '07 ($p > h$), B-Nakano-Pillen '16 ($p > 3$ or 5)
  - More cases ...
  - Key Difference: some 2-dimensional cases
  - AR used LHS and reduction to B_1
  - BNP used computations for B_r and inverse limits

Andersen-Jantzen '84 (and Kumar-Lauritzen-Thomsen '99):

$$\begin{aligned} \mathrm{H}^i(G_1, H^0(\lambda))^{(-1)} &\cong \mathrm{ind}_B^G\left(\mathrm{H}^i(B_1,\lambda)^{(-1)}\right) \\ &\cong \begin{cases} \mathrm{ind}_B^G\left(S^{\frac{i-\ell(w)}{2}}(\mathfrak{u}^*)\otimes\mu\right) & \text{if } \lambda = w\cdot 0 + p\mu, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

Note: Hidden is a computation of $\mathrm{H}^i(B_1, \lambda)$

Questions:

- Small primes?
- G_r and B_r for higher r?

Ngo '13: $G = SL_2$ - $\mathrm{H}^i(G_r, H^0(\lambda))$ and $\mathrm{H}^i(B_r, \lambda)$

An Inductive/Reductive Approach

- Recall the Induction Problem: When is

$$\mathrm{H}^i(G_r, H^0(\lambda))^{(-1)} \cong \mathrm{ind}_B^G \left(\mathrm{H}^i(B_r, \lambda)^{(-1)}\right)?$$

- LHS for $B_1 \trianglelefteq B_r$ can be used to inductively push B_1-results to B_r-results
- $\mathrm{H}^i(B_1, \lambda) \cong \left(\mathrm{H}^i(U_1, \lambda)^{T_1}\right) \cong \left(\mathrm{H}^i(U_1, k) \otimes \lambda\right)^{T_1}$
- Use the spectral sequence

$$E_2^{2i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes \mathrm{H}^j(\mathfrak{u}, k) \Rightarrow \mathrm{H}^{2i+j}(U_1, k)$$

- Need $\mathrm{H}^j(\mathfrak{u}, k)$

Answers

Degree 1:

- Jantzen '91 - $\mathrm{H}^1(G_1, H^0(\lambda))$, $\mathrm{H}^1(B_1, \lambda)$, and some $\mathrm{H}^1(G_1, L(\lambda))$
- B-Nakano-Pillen '04 - extended to $r > 1$

Degree 2:

- B-Nakano-Pillen '07 - $\mathrm{H}^2(G_r, H^0(\lambda))$ and $\mathrm{H}^2(B_r, \lambda)$ for $p \geq 3$
- Wright '11 - $p = 2$

Degree 3:

- B-Nakano-Pillen '16 - $\mathrm{H}^3(G_r, H^0(\lambda))$ and $\mathrm{H}^3(B_r, \lambda)$ for $p \geq 5, 7$
- Small primes still open

As noted above, this approach needs computations of $\mathrm{H}^i(\mathfrak{u}, k)$.

Further, as the degree increases, an inductive approach (for higher r) needs other Lie algebra cohomology computations:

$$\mathrm{H}^i(\mathfrak{u}, \mathfrak{u}^*)$$

or more generally

$$\mathrm{H}^i(\mathfrak{u}, S^j(\mathfrak{u}^*))$$

Note: $\mathrm{H}^i(\mathfrak{u}, \mathfrak{u}^*)$ is similar to $\mathrm{H}^{i+1}(\mathfrak{u}, k)$

Noted Earlier: Kostant's Theorem for $\mathrm{H}^i(\mathfrak{u}, L(\lambda))$ for $p \geq h - 1$ and "small" λ

Other modules?

$G(\mathbb{F}_q)$-Cohomology of Simple Modules

$\mathrm{H}^i(G(\mathbb{F}_q), L(\lambda)) =?$

Computations in degrees 1 and 2 for "small" weights. E.g., weights less than or equal to a fundamental dominant weight

- Cline-Parshall-Scott '75, '77
- Jones '75
- Avrunin '78
- Bell '78
- Kleschev '94
- UGA Vigre Algebra Group '12, '13

Self-Extensions

Consider $\mathrm{Ext}^i_G(M, M)$, particularly $\mathrm{Ext}^1_G(M, M)$:

$$0 \to M \to N \to M \to 0$$

For simple G-modules,

$$\mathrm{Ext}^1_G(L(\lambda), L(\lambda)) = 0$$

Can use earlier fact:

$$\mathrm{Ext}^1_G(L(\lambda), L(\lambda)) \cong \mathrm{Hom}_G(L(\lambda), H^0(\lambda)/L(\lambda)) = 0,$$

since $H^0(\lambda)/L(\lambda)$ contains no copies of $L(\lambda)$.

Self-Extensions for G_r and $G(\mathbb{F}_q)$

Andersen '84: $\mathrm{Ext}^1_{G_r}(L(\lambda), L(\lambda)) = 0$ for $\lambda \in X(T)_+$, except in Type C_n when $p = 2$

For example, $\mathrm{H}^1(G_1, k) \neq 0$ in that case.

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For example, $\mathrm{H}^1(G_1, k) \neq 0$ in that case.

For $G(\mathbb{F}_q)$, the answer differs for $r = 1$ vs. $r > 1$ and remains open in small primes.

B-Nakano-Pillen '04: For $p \geq 3(h-1)$, $r \geq 2$, and $\lambda, \mu \in X_r(T)$.

- If $\lambda_{r-1} = \mu_{r-1}$, then $\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\lambda), L(\mu))$.
- In particular, $\mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\lambda)) = 0$.

This uses the above G_r result.

Self-Extensions for $G(\mathbb{F}_p)$

Let $p \geq 3(h-1)$ and $\lambda \in X_1(T)$. If either

(a) G does not have underlying root system of type A_1 or C_n or

(b) $\langle \lambda, \alpha_n^\vee \rangle \neq \frac{p-2-c}{2}$, where α_n is the long simple root and c is an odd integer with $0 < |c| \leq h-1$,

then $\mathrm{Ext}^1_{G(\mathbb{F}_p)}(L(\lambda), L(\lambda)) = 0$.

Pillen '06: constructed classes in the excluded cases for all ranks and all odd primes

Bounding Dimensions

Longstanding question for finite simple groups:

How large can the first cohomology of a simple module be?

Bounding Dimensions

Longstanding question for finite simple groups:

How large can the first cohomology of a simple module be?

Main case: finite groups of Lie type

For many years thought to be small:

- 2?
- then 3 - Scott '03
- until AIM Workshop in 2012 ...
- Very large examples - computations of Lübeck and Scott-Sprowl '13

Bounds

Cline-Parshall-Scott '09: There is a bound depending only on the root system

$$\dim \mathrm{H}^1(G(\mathbb{F}_q), L(\lambda)) \leq C(\Phi)$$

Parshall-Scott 'II: Existence of bound for $\dim \mathrm{Ext}^1_{G(\mathbb{F}_q)}(L(\lambda), L(\mu))$

Guralnick-Tiep '11: Bounds in the non-defining (or cross-charateristic case)

Parker-Stewart '14: Precise bounds and rate of growth of the max dimension relative to the rank

Bounds for G-extensions

The $G(\mathbb{F}_q)$ results were generally obtained by relating $G(\mathbb{F}_q)$-Extensions to G-Extensions and knowing bounds there.

Parshall-Scott '11: Existence of a bound on $\dim \mathrm{H}^i(G, L(\lambda))$

- Depending on Φ and i
- Also an Ext-version

Stewart '12: Existence of simple SL_2-modules with $\dim \mathrm{H}^{2n}(SL_2, L) \geq 2^{n-1}$

- i.e., exponential growth in the degree

Back to $G(\mathbb{F}_q)$ and G_r

B-Nakano-Parshall-Pillen-Scott-Stewart '15: Existence of a bound on $\dim \mathrm{H}^i(G(\mathbb{F}_q), L(\lambda))$

- Depending on Φ and i
- Also an Ext-version
- Also a version for Frobenius kernels - G_r
- For G_r - existence of cohomology which grows with the rank

Still open question to illustrate precise bounds.

Bounding in Terms of Dimensions

Many results, for G or $G(\mathbb{F}_q)$ or finite groups in general, of the form

$$\dim \mathrm{H}^i(G, M) \leq C \cdot \dim M$$

For example, for quasi-simple finite groups

- Guralnick-Hoffman '98: Degree 1 and M irreducible
    - $C = 1/2$
- Guralnick-Kantor-Kassabov-Lubotzky '07: Degree 2
    - $C = 17.5$

More Bounds

- AIM Group: B-Boe-Drupieski-Nakano-Parshall-Pillen-Wright '14
  - For B- and G-cohomology - arbiitrary modules
  - Degrees 1, 2: $C = 1$
  - Degree 3 with $p > h$: $C = 2$
  - For $G(\mathbb{F}_q)$ and sufficiently large r - arbitrary modules
  - Degree 1: $C = 1/h$
  - Degree 2: $C = 1$
  - Degree 3 with $p > h$: $C = 2$

Again, open questions remain ...

The End

Bottom-Line: There are many, many questions out there waiting to be answered ...

Thank you!