On the affine VW supercategory

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Joint with
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A continuation of Iva Halacheva's talk.

But some background will provided.

Background: vector superspaces. Work over $\mathbb{C}$.

A $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is a *vector superspace*.

The superdimension of V is
$\dim(V) := (\dim V_{\overline{0}} | \dim V_{\overline{1}}) = \dim V_{\overline{0}} - \dim V_{\overline{1}}$.

Given a homogeneous element $v \in V$, the *parity* (or the *degree*) of v is $\overline{v} \in \{\overline{0}, \overline{1}\}$.
The parity switching functor π sends $V_{\overline{0}} \mapsto V_{\overline{1}}$ and $V_{\overline{1}} \mapsto V_{\overline{0}}$.

Let $m = \dim V_{\overline{0}}$ and $n = \dim V_{\overline{1}}$. The Lie superalgebra is $\mathfrak{gl}(m|n) := \mathrm{End}_{\mathbb{C}}(V)$.

That is, given a homogeneous ordered basis for V:

$$V = \underbrace{\mathbb{C}\{v_1, \ldots, v_m\}}_{V_{\overline{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \ldots, v_{n'}\}}_{V_{\overline{1}}},$$

Matrix representation for $\mathfrak{gl}(m|n)$.

the *Lie superalgebra* is the endomorphism algebra

$$\mathfrak{gl}(m|n) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A \in M_{m,m}, B, C^t \in M_{m,n}, D \in M_{n,n} \right\},$$

where $M_{i,j} := M_{i,j}(\mathbb{C})$. Since $\mathfrak{gl}(m|n) = \mathfrak{gl}(m|n)_{\overline{0}} \oplus \mathfrak{gl}(m|n)_{\overline{1}}$,

$$\mathfrak{gl}(m|n)_{\overline{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\} \text{ and } \mathfrak{gl}(m|n)_{\overline{1}} = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right\}.$$

We say V is the *natural representation* of $\mathfrak{gl}(m|n)$.

The grading on $\mathfrak{gl}(m|n)$ is induced by V, with *Lie superbracket* (supercommutator) $[x, y] = xy - (-1)^{\overline{x}\overline{y}} yx$ for x, y homogeneous.

Periplectic Lie superalgebras $\mathfrak{p}(n)$.

Let $m = n$. Then

$$V = \mathbb{C}^{2n} = \underbrace{\mathbb{C}\{v_1, \ldots, v_n\}}_{V_{\overline{0}}} \oplus \underbrace{\mathbb{C}\{v_{1'}, \ldots, v_{n'}\}}_{V_{\overline{1}}}.$$

Define $\beta : V \otimes V \to \mathbb{C}$ as a symmetric, odd, nondegenerate bilinear form satisfying:

$$\beta(v, w) = \beta(w, v), \qquad \beta(v, w) = 0 \quad \text{if } \overline{v} = \overline{w}.$$

We define *periplectic (strange) Lie superalgebras* as:

$$\mathfrak{p}(n) := \{x \in \mathrm{End}_{\mathbb{C}}(V) : \beta(xv, w) + (-1)^{\overline{xv}}\beta(v, xw) = 0\}.$$

In terms of above basis,

$$\mathfrak{p}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n|n) : B = B^t, C = -C^t \right\}.$$

Symmetric monoidal structure.

Consider the category $\mathcal{C}$ of representations of $\mathfrak{p}(n)$ with

$$\mathrm{Hom}_{\mathfrak{p}(n)}(V, V') := \{f : V \to V' : f \text{ homogeneous}, \mathbb{C}\text{-linear}, \\ f(x.v) = (-1)^{\overline{xf}} x.f(v), v \in V, x \in \mathfrak{p}(n)\}.$$

Then $U(\mathfrak{p}(n))$ of $\mathfrak{p}(n)$ is a Hopf superalgebra:

- (coproduct) $\Delta(x) = x \otimes 1 + 1 \otimes x$,
- (counit) $\epsilon(x) = 0$,
- (antipode) S(x) = -x.

So the category of representations of $\mathfrak{p}(n)$ is monoidal.
For $x \otimes y \in U(\mathfrak{p}(n)) \otimes U(\mathfrak{p}(n))$ on $v \otimes w$,

$$(x \otimes y).(v \otimes w) = (-1)^{\overline{yv}} xv \otimes yw.$$

Symmetric monoidal structure.

For $x, y, a, b \in U(\mathfrak{p}(n))$,

$$(x \otimes y) \circ (a \otimes b) := (-1)^{\overline{ya}} (x \circ a) \otimes (y \circ b),$$

and for two representations V and V', the *super swap*

$$\sigma : V \otimes V' \longrightarrow V' \otimes V, \quad \sigma(v \otimes w) = (-1)^{\overline{vw}} w \otimes v$$

is a map of $\mathfrak{p}(n)$-representations satisfying $\sigma^* = -\sigma$.
Thus $\mathcal{C}$ is a symmetric monoidal category.
Furthermore, β induces a representation V and its dual V^* via

$$V \to V^*, \quad v \mapsto \beta(v, -),$$

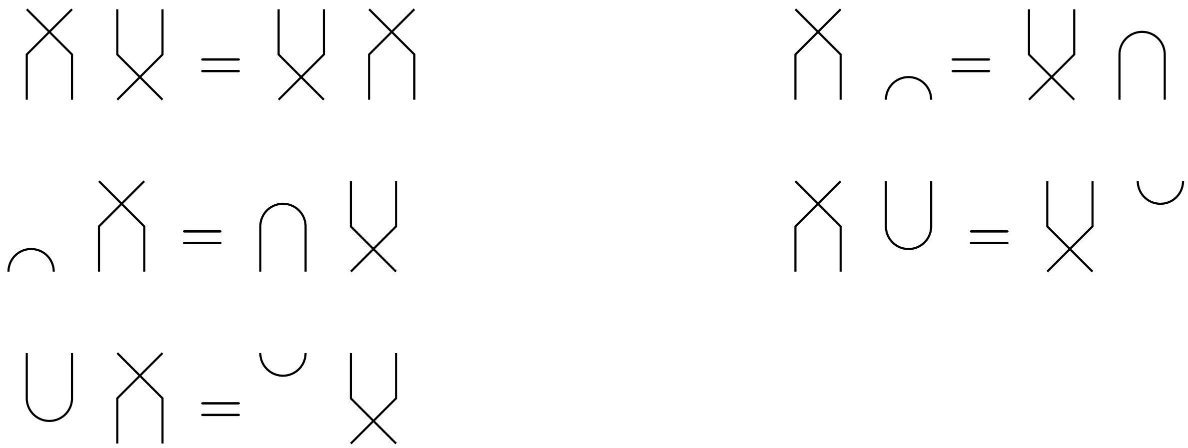
identifying $V_{\overline{1}}$ with $V_{\overline{0}}^*$ and $V_{\overline{0}}$ with $V_{\overline{1}}^*$. This induces the dual map

$$\beta^* : \mathbb{C} \cong \mathbb{C}^* \longrightarrow (V \otimes V)^* \cong V \otimes V, \quad \beta^*(1) = \sum_i -v_i \otimes v_{i'} + v_{i'} \otimes v_i,$$

where $\overline{\beta} = \overline{\beta^*} = 1$.

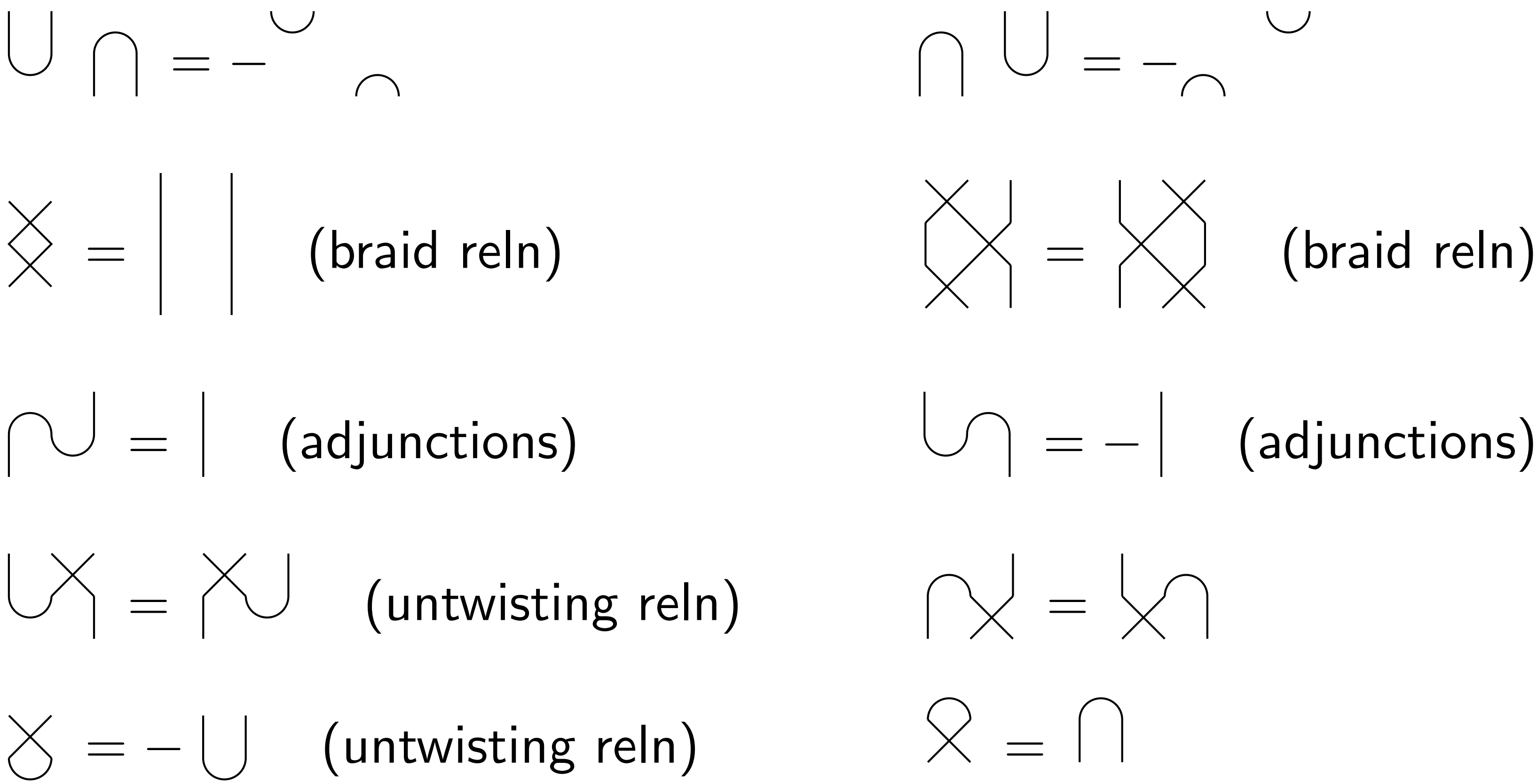
Affine Brauer algebras (generators and local moves).

$s\mathbb{W}_a$ has generators s_i, b_i, b_i^*, y_j, where $i = 1, \ldots, a-1$, $j = 1, \ldots, a$ and relations

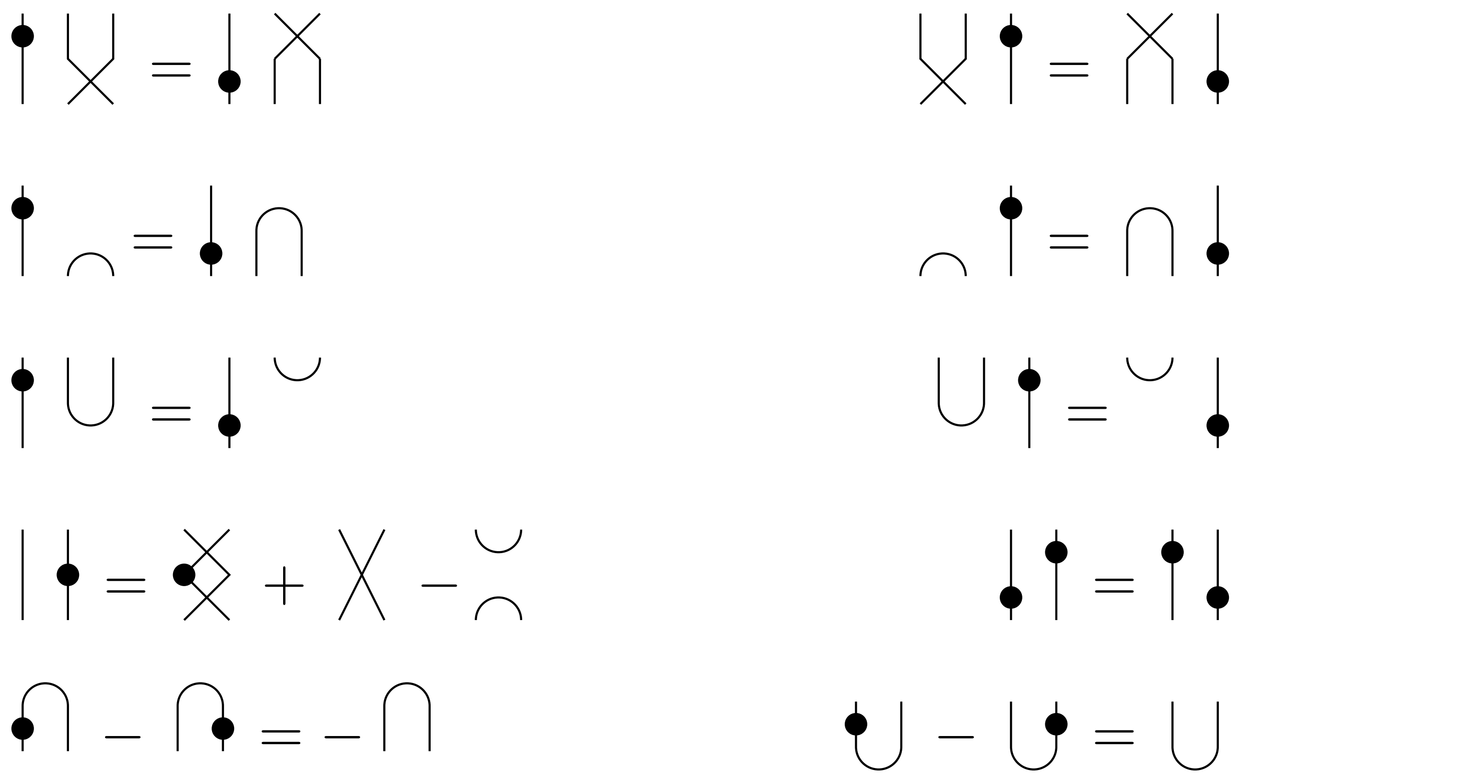


Continued in the next slide.

Affine Brauer algebras (local moves; continued).



Affine Brauer algebras (local moves; continued).



Connectors.

Each normal diagram $d \in \mathrm{Hom}_{s\mathcal{B}r}(a, b)$, where $a, b \in \mathbb{N}_0$, gives rise to a partition $P(d)$ of the set of $a + b$ points into 2-element subsets given by the endpoints of the strings in the diagram d.

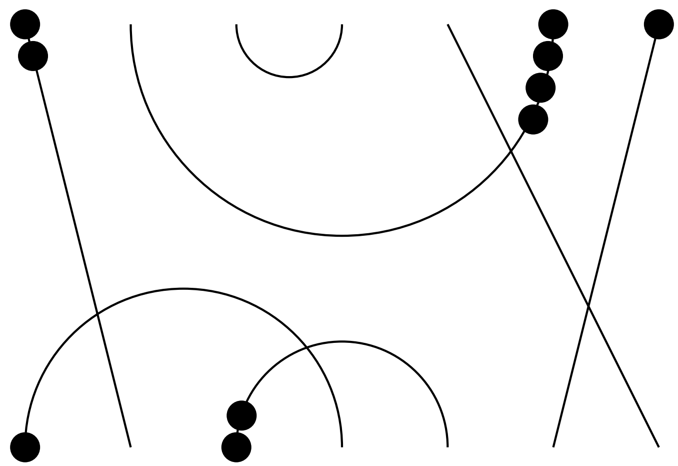
We call such a partition a *connector*, and write $\mathrm{Conn}(a, b)$ to denote the set of all such connectors.

For each connector $c \in \mathrm{Conn}(a, b)$, we pick a normal diagram $d_c \in P^{-1}(c) \subset \mathrm{Hom}_{s\mathcal{B}r}(a, b)$.

Remark. Different normal diagrams in a single fibre $P^{-1}(c)$ differ only by braid relations, and thus represent the same morphism.

(Regular) monomials and (normal) diagrams.

An example.



Algebraically, it is written as $y_1^2 y_6^4 y_7 s_5 b_2^* b_2 b_4^* b_4 s_1 s_3 s_6 y_1 y_3^2$.

Our affine VW superalgebra $s\mathbb{W}_a$ is:

- super (signed) version of the degenerate BMW algebra,
- the signed version of the affine VW algebra, and
- an affine version of the Brauer superalgebra.

Theorem

The set $S_{a,b} = \{d_c : c \in \mathrm{Conn}(a,b)\}$ *is a basis of* $\mathrm{Hom}_{s\mathcal{B}r}(a,b)$.

Let $S^{\bullet}_{a,b}$ be the set of normal dotted diagrams obtained by taking all diagrams in $S_{a,b}$ and adding dots to them in all possible ways.

Let $S^{k}_{a,b} \subset S^{\bullet}_{a,b}$ and $S^{\leq k}_{a,b} = \bigcup_{l=0}^{k} S^{l}_{a,b}$ be the sets of such diagrams with exactly k dots and at most k dots, respectively.

When $k = 0$, $S^{0}_{a,b} = S^{\leq 0}_{a,b} = S_{a,b}$.

Theorem (Basis theorem)

The set $S^{\leq k}_{a,b}$ *is a basis of* $\mathrm{Hom}_{s\mathbb{W}}(a,b)^{\leq k}$, *and consequently the set* $S^{\bullet}_{a,b}$ *is a basis of* $\mathrm{Hom}_{s\mathbb{W}}(a,b)$.

The center of $s\mathbb{W}_a = \mathrm{End}_{s\mathbb{W}}(a)$, $a \geq 2 \in \mathbb{N}$.

Theorem

The center $Z(s\mathbb{W}_a)$ consists of all polynomials of the form

$$\prod_{1\leq i<j\leq a}((y_i - y_j)^2 - 1)\widetilde{f} + c,$$

where $\widetilde{f} \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}$ and $c \in \mathbb{C}$.

The deformed squared Vandermonde determinant $\prod_{1\leq i<j\leq a}((y_i - y_j)^2 - 1)$ is symmetric, so

$$\prod_{1\leq i<j\leq a}((y_i - y_j)^2 - 1) \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}.$$

The algebra $A_\hbar$ and its specializations A_t, where $t \in \mathbb{C}$.

Definition

Let $A_\hbar$ be the superalgebra over $\mathbb{C}[\hbar]$ with generators $s_i,\ e_i,\ y_j$ for $1 \le i \le a-1,\ 1 \le j \le a$, where $\overline{s_i} = \overline{e_i} = \overline{y_j} = 0$, subject to the relations:

1. Involutions: $s_i^2 = 1$ for $1 \le i < a$.
2. Commutation relations:
   - 2.1 $s_i e_j = e_j s_i$ if $|i-j| > 1$,
   - 2.2 $e_i e_j = e_j e_i$ if $|i-j| > 1$,
   - 2.3 $e_i y_j = y_j e_i$ if $j \neq i, i+1$,
   - 2.4 $y_i y_j = y_j y_i$ for $1 \le i, j \le a$.
3. Affine braid relations:
   - 3.1 $s_i s_j = s_j s_i$ if $|i-j| > 1$,
   - 3.2 $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $1 \le i \le a-1$,
   - 3.3 $s_i y_j = y_j s_i$ if $j \neq i, i+1$.
4. Snake relations:
   - 4.1 $e_{i+1} e_i e_{i+1} = -e_{i+1}$,
   - 4.2 $e_i e_{i+1} e_i = -e_i$ for $1 \le i \le a-2$.
5. Tangle and untwisting relations:
   - 5.1 $e_i s_i = e_i$ and $s_i e_i = -e_i$ for $1 \le i \le a-1$,
   - 5.2 $s_i e_{i+1} e_i = s_{i+1} e_i$,
   - 5.3 $s_{i+1} e_i e_{i+1} = -s_i e_{i+1}$,
   - 5.4 $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$,
   - 5.5 $e_i e_{i+1} s_i = -e_i s_{i+1}$ for $1 \le i \le a-2$.
6. Idempotent relations: $e_i^2 = 0$ for $1 \le i \le a-1$.
7. Skein relations:
   - 7.1 $s_i y_i - y_{i+1} s_i = -\hbar e_i - \hbar$,
   - 7.2 $y_i s_i - s_i y_{i+1} = \hbar e_i - \hbar$ for $1 \le i \le a-1$.
8. Unwrapping relations: $e_1 y_1^k e_1 = 0$ for $k \in \mathbb{N}$.
9. (Anti)-symmetry relations:
   - 9.1 $e_i(y_{i+1} - y_i) = \hbar e_i$,
   - 9.2 $(y_{i+1} - y_i) e_i = -\hbar e_i$ for $1 \le i \le a-1$.

For $t \in \mathbb{C}$, let A_t be the quotient of $A_\hbar$ by the ideal generated by $\hbar - t$.

A sketch of proof of the Theorem on page 15.

1. The filtered algebra $s\mathbb{W}_a$ (via the filtration by the degree of the polynomials in $\mathbb{C}[y_1, \dots, y_a]$) is a Poincaré-Birkhoff-Witt (PBW) deformation of the associated graded superalgebra $gs\mathbb{W}_a = \mathrm{gr}(s\mathbb{W}_a)$,
2. For $\hbar$ a parameter, the Rees construction gives the algebra $A_\hbar$ over $\mathbb{C}[\hbar]$ such that the specializations $\hbar = 1$ and $\hbar = 0$ are precisely $A_1 = s\mathbb{W}_a$ and $A_0 = gs\mathbb{W}_a$,
3. Describe the center of the $\mathbb{C}[\hbar]$-algebra $A_\hbar$, and all its specializations A_t for any $t \in \mathbb{C}$ using the Basis Theorem,
4. Determine the center of $gs\mathbb{W}_a$ using the isomorphism $\mathrm{Rees}(Z(A_1)) \cong Z(\mathrm{Rees}(A_1)) \cong Z(A_\hbar)$, and
5. Find a lift of the appropriate basis elements to $s\mathbb{W}_a$ to obtain the center of $s\mathbb{W}_a$.

Expanding on 2.

Let $B = \bigcup_{k \geq 0} B^{\leq k}$ be a filtered $\mathbb{C}$-algebra. The *Rees algebra* of B is the $\mathbb{C}[\hbar]$-algebra $\text{Rees}(B)$, given as a $\mathbb{C}$-vector space by $\text{Rees}(B) = \bigoplus_{k \geq 0} B^{\leq k} \hbar^k$, with multiplication and the $\hbar$-action given by

$$(a\hbar^i)(b\hbar^j) = (ab)\hbar^{i+j} \text{ for } a \in B^{\leq i}, b \in B^{\leq j}, \text{ and } ab \in B^{\leq i+j},$$

the product in B. It is graded as a $\mathbb{C}$-algebra by the powers of $\hbar$.

Lemma

1. *Let $\bigcup_{i \geq 0} S_i$ be a basis of B compatible with the filtration, where S_i's are pairwise disjoint, and $\bigcup_{i=0}^{k} S_i$ is a basis of $B^{\leq k}$. Then $\bigcup_{i \geq 0} S_i \hbar^i$ is a $\mathbb{C}[\hbar]$-basis of* $\text{Rees}(B)$.
2. $Z(\text{Rees}(B)) = \text{Rees}(Z(B))$.
3. $\text{Rees}(A_1) \cong A_\hbar$, *an isomorphism of $\mathbb{C}[\hbar]$-algebras.*

Expanding on 3.

Show that $Z(A_\hbar) \subseteq \mathbb{C}[\hbar][y_1, \ldots, y_a]^{S_a}$.

Lemma

For $f \in A_\hbar$, the following are equivalent:

(a) $fy_i = y_i f$ *for all* $i \in [a] = \{1, 2, \ldots, a\}$;

(b) $f \in \mathbb{C}[\hbar][y_1, \ldots, y_a]$.

So $Z(A_\hbar) \subseteq \mathbb{C}[\hbar][y_1, \ldots, y_a]$.

Lemma

Let $f \in \mathbb{C}[\hbar][y_1, \ldots, y_a] \subseteq A_\hbar$ *and* $1 \leq i \leq a - 1$.

(a) *If* $fs_i = s_i f$, *then* $f(y_1, \ldots, y_i, y_{i+1}, \ldots, y_a) = f(y_1, \ldots, y_{i+1}, y_i, \ldots, y_a)$.

(b) *For the special value* $\hbar = 0$, *the converse also holds: if* $f(y_1, \ldots, y_i, y_{i+1}, \ldots, y_a) = f(y_1, \ldots, y_{i+1}, y_i, \ldots, y_a)$, *then* $fs_i = s_i f$ *in* A_0.

So $Z(A_\hbar)$ is a subalgebra of $\mathbb{C}[\hbar][y_1, \ldots, y_a]^{S_a}$.

Expanding on 3 (*continued*).

Consider the following elements in $\mathbb{C}[\hbar][y_1, \dots, y_a]$:

$$z_{ij} = (y_i - y_j)^2, \text{ for } 1 \leq i \neq j \leq a \quad \text{and} \quad D_\hbar = \prod_{1 \leq i < j \leq a} (z_{ij} - \hbar^2),$$

where $D_\hbar$ is symmetric. So $D_\hbar \in \mathbb{C}[\hbar][y_1, \dots, y_a]^{S_a}$.
Use $D_\hbar$ to produce central elements in $A_\hbar$.

Lemma

1. *For any $1 \leq i \leq a-1$, $e_i \cdot (z_{i,i+1} - \hbar^2) = (z_{i,i+1} - \hbar^2) \cdot e_i = 0$ in $A_\hbar$, and consequently $e_i D_\hbar = D_\hbar e_i = 0$.*
2. *For any $1 \leq k \leq a-1$, we have $D_\hbar s_k = s_k D_\hbar$.*
3. *Let $1 \leq i \leq a-1$, and let $\tilde{f} \in \mathbb{C}[\hbar][y_1, \dots, y_a]$ be symmetric in y_i, y_{i+1}. Then there exist polynomials $p_j = p_j(y_1, \dots, y_a) \in \mathbb{C}[\hbar][y_1, \dots, y_a]$ such that $\tilde{f} s_i = s_i \tilde{f} + \sum_{j=0}^{\deg \tilde{f} - 1} y_i^j \cdot e_i \cdot p_j$.*

Expanding on 3 (*continued*).

Lemma

Let $\tilde{f} \in \mathbb{C}[\hbar][y_1, \ldots, y_a]^{S_a}$ be an arbitrary symmetric polynomial, and $c \in \mathbb{C}$. Then $f = D_\hbar \tilde{f} + c \in Z(A_\hbar)$.

Expanding on 4.

Proposition. The center $Z(A_0)$ of the graded VW superalgebra gsW_a consists of all $f \in \mathbb{C}[y_1, \ldots, y_a]$ of the form $f = D_0 \tilde{f} + c$, for $\tilde{f} \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}$ and $c \in \mathbb{C}$.

Expanding on 5.

Theorem

The center $Z(s\mathbb{W}_a)$ of the VW superalgebra $s\mathbb{W}_a = A_1$ consists of all $f \in \mathbb{C}[y_1, \ldots, y_n]$ of the form $f = D_1\tilde{f} + c$, for an arbitrary symmetric polynomial $\tilde{f} \in \mathbb{C}[y_1, \ldots, y_a]^{S_a}$ and $c \in \mathbb{C}$.

Proof.

For any filtered algebra B there exists a canonical injective algebra homomorphism $\varphi : \mathrm{gr}\, Z(B) \hookrightarrow Z(\mathrm{gr}(B))$, given by $\varphi(f + Z(B)^{\leq(k-1)}) = f + B^{\leq(k-1)}$ for $f \in Z(B)^{\leq k}$. For $B = s\mathbb{W}_a$ and $\mathrm{gr}(B) = gs\mathbb{W}_a$, $Z(A_0)$ consists of elements of the form $f = D_0\tilde{f} + c$ for $\tilde{f}$ a symmetric polynomial and c a constant. Since $D_1\tilde{f} + c \in Z(s\mathbb{W}_a)$, we have $\varphi(c) = c$, and for $\tilde{f}$ symmetric and homogeneous of degree k, $\varphi(D_1\tilde{f} + s\mathbb{W}_a^{\leq a(a-1)+k-1}) = D_0\tilde{f}$. Using the above Proposition, we see that every $f \in Z(gs\mathbb{W}_a)$ is in the image of φ, so φ is an isomorphism. $\square$

Expanding on 5 (*continued*).

Theorem

The center $Z(A_\hbar)$ of the superalgebra $A_\hbar$ consists of polynomials $f \in \mathbb{C}[\hbar][y_1, \dots, y_n]$ of the form $f = D_\hbar \tilde{f} + c$, for an arbitrary symmetric polynomial $\tilde{f} \in \mathbb{C}[\hbar][y_1, \dots, y_a]^{S_a}$ and $c \in \mathbb{C}[\hbar]$.

Proof.

The center $Z(A_\hbar)$ is isomorphic to $Z(\mathrm{Rees}(A_1))$, which is also isomorphic to $\mathrm{Rees}(Z(A_1))$. The center $Z(A_1)$ consists of elements of the form $f = D_1\tilde{f} + c$, with $\tilde{f} \in \mathbb{C}[y_1, \dots, y_a]^{S_a}$ and $c \in \mathbb{C}$. Assume $\tilde{f}$ is homogeneous of degree k. Then $D_1\tilde{f} \in A_1^{\leq k+a(a-1)}$, which gives an element $D_1\tilde{f}\hbar^{k+a(a-1)}$ of $\mathrm{Rees}(Z(A_1)) \cong Z(\mathrm{Rees}(A_1))$. We see that $Z(A_\hbar)$ is spanned by constants and the preimages under the isomorphism $A_\hbar \cong \mathrm{Rees}(A_1)$ of elements $D_1\tilde{f}\hbar^{k+a(a-1)}$, which are equal to $D_\hbar\tilde{f}$. □

Thank you.

Questions?