Reductive subgroup schemes of a parahoric group scheme

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Contents

Outline

Levi decompositions / Levi factors

Let H be a conn linear alg group over a field F of char. $p \geq 0$.

- *Assumption (R):* s'pose unip radical $R = R_u H$ defined over F.
- (R) fails for $R_{E/F}\mathbf{G}_m$ if E purely insep of deg $p > 0$ over F.
- (R) always holds when F is perfect.
- A closed subgroup $M \subset H$ is a *Levi factor* if $\pi_{|M} : M \to H/R$ is an isomorphism of algebraic groups.
- If $p = 0$, H always has a *Levi decomposition*, but it need not when $p > 0$ (examples to follow...).

Groups with no Levi factor (part 1)

Assume $p > 0$ and let $W_{2/F} = W(F)/p^2W(F)$ be the ring of length 2 Witt vectors over F. Let $\mathcal{G}$ a split semisimple group scheme over $\mathbf{Z}$. There is a linear alg F-group H with the following properties:

- $H(F) = \mathcal{G}(W_{2/F})$
- There is a non-split sequence

$$0 \to \mathrm{Lie}(\mathcal{G}_F)^{[1]} \to H \to \mathcal{G}_F \to 1$$

  hence H satisfies (R) and has no Levi factor.
- (The “exponent” [1] just indicates that the usual adjoint action of $\mathcal{G}_F$ is “twisted” by Frobenius).

Groups with no Levi factor (part 2)

Recall (R) is in effect. S'pose in addition that there is an H-equivariant isomorphism $R \simeq \mathrm{Lie}(R) = V$ of algebraic groups.

- Consider the (strictly) exact sequence

$$0 \to V \to H \xrightarrow{\pi} G \to 1$$

  where $G = H/R$ is the reduc quotient.
- Since V is split unip, result of Rosenlicht guarantees that π has a *section*; i.e. $\exists$ regular $\sigma : G \to H$ with $\pi \circ \sigma = 1_G$
- Use σ to build 2-cocycle α_H via

$$\alpha_H = \left((x, y) \mapsto \sigma(xy)^{-1}\sigma(x)\sigma(y)\right) : G \times G \to V$$

Proposition

H has a Levi factor if and only if $[\alpha_H] = 0$ in $H^2(G, V)$ where $H^2(G, V)$ is the Hochschild cohomology group.

Groups with no Levi factor (conclusion)

Remark

If G is reductive in char. p, combining the above constructions shows that $H^2(G, \mathrm{Lie}(G)^{[1]}) \neq 0$.

Remark

If

$$0 \to V \to H \to G \to 1$$

is a split extension, where V is a lin repr of G, then $H^1(G, V)$ describes the $H(F)$-conjugacy classes of Levi factors of H.

Outline

Preliminaries

- Let K be the field of fractions of a complete DVR $\mathcal{A}$ with residue field $\mathcal{A}/\pi\mathcal{A} = \mathrm{k}$.
- e.g. $\mathcal{A} = W(\mathrm{k})$ ("mixed characteristic"), or $\mathcal{A} = \mathrm{k}[\![t]\!]$ ("equal characteristic").
- Let G be a connected and reductive group over K.
- The parahoric group schemes attached to G are certain affine, smooth group schemes $\mathcal{P}$ over $\mathcal{A}$ having generic fiber $\mathcal{P}_{\mathrm{K}} = G$.
- If e.g. G is split over K, there is a split reductive group scheme $\mathcal{G}$ over $\mathcal{A}$ with $G = \mathcal{G}_{\mathrm{K}}$, and $\mathcal{G}$ is a parahoric group scheme.
- But in general, parahoric group schemes $\mathcal{P}$ are *not* reductive over $\mathcal{A}$, even for split G. In particular, the special fiber $\mathcal{P}_{\mathrm{k}}$ need not be a reductive k-group.

Example: stabilizer of a lattice flag

Let $G = \mathrm{GL}(V)$ and let $\pi\mathcal{L} \subset \mathcal{M} \subset \mathcal{L}$ be a flag of $\mathcal{A}$-lattices in V.

- View $G \times G$ as the generic fiber of $\mathcal{H} = \mathrm{GL}(\mathcal{L}) \times \mathrm{GL}(\mathcal{M})$.
- Denote by Δ the diagonal copy of G in $G \times G$.
- Let $\mathcal{P}$ be the schematic closure of Δ in $\mathcal{H}$.
- Then $\mathcal{P}$ is a parahoric group scheme, and it "is" the stabilizer of the given lattice flag.
- The special fiber $\mathcal{P}_{\mathrm{k}}$ has reductive quotient

$$\mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \quad \text{where} \quad W_1 = \mathcal{L}/\mathcal{M} \quad W_2 = \mathcal{M}/\pi\mathcal{L}$$

and

$$R_u(\mathcal{P}_{\mathrm{k}}) = \mathrm{Hom}_{\mathrm{k}}(W_1, W_2) \oplus \mathrm{Hom}_{\mathrm{k}}(W_2, W_1).$$

Unipotent radical of the special fiber of $\mathcal{P}$

S'pose G splits over an unramif ext $\mathrm{L} \supset \mathrm{K}$. Concerning (R):

Proposition

Suppose that G splits over an unramified extension of K, and let $\mathcal{P}$ be a parahoric group scheme attached to G. Then $R_u\mathcal{P}_{\mathrm{k}}$ is defined and split over k.

- Maybe worth saying when k may not be pefect: $\mathrm{L} \supset \mathrm{K}$ unramified *requires* the residue field extension $\ell \supset \mathrm{k}$ to be separable.
- Idea of the proof: immediately reduce to the case of split G. Write $\mathcal{A}_0 = \mathbf{Z}_p$ or $\mathbf{F}_p((t))$ and write $\mathrm{K}_0 = \mathrm{Frac}(\mathcal{A}_0)$. Then G and $\mathcal{P}$ arise by base change from G_0 and $\mathcal{P}_0$ over K_0 and $\mathcal{A}_0$. And the residue field of $\mathcal{A}_0$ is of course perfect.

Outline

Levi factors of the special fiber of a parahoric

Question: let $\mathcal{P}$ be a parahoric group scheme attached to G.
When does the special fiber $\mathcal{P}_{\mathrm{k}}$ have a Levi decomposition?
For the following two Theorems, suppose that k is perfect.

Theorem (McNinch 2010)

Suppose that G splits over an unramified extension of K. Then $\mathcal{P}_{\mathrm{k}}$ has a Levi factor. Moreover:

(a) *If G is split, each maximal split k-torus of $\mathcal{P}_{\mathrm{k}}$ is contained in a unique Levi factor. In particular, Levi factors are $\mathcal{P}(\mathrm{k})$-conjugate.*

(b) *Levi factors of $\mathcal{P}_{\mathrm{k}}$ are geometrically conjugate.*

Theorem (McNinch 2014)

Suppose that G splits over a tamely ramified extension of K. $\mathcal{P}_{\overline{\mathrm{k}}}$ has a Levi factor, where $\overline{\mathrm{k}}$ is an algebraic (=separable) closure of k.

Example: Non-conjugate Levis of some $\mathcal{P}_{\mathrm{k}}$

S'pose char p of k is $\neq 2$. Let V be a vector space of dimension $2m$ over a quadratic ramified ext $\mathrm{L} \supset \mathrm{K}$ and equip V with a "quasi-split" hermitian form h. Put $G = \mathrm{SU}(V, h)$.

- There is an $\mathcal{A}_{\mathrm{L}}$-lattice $\mathcal{L} \subset V$ such that h determines nondeg sympl form on the k-vector space $M = \mathcal{L}/\pi_{\mathrm{L}}\mathcal{L}$.
- $\mathcal{L}$ determines a parahoric $\mathcal{P}$ for G for which $\exists$ exact seq

$$0 \to W \to \mathcal{P}_{\mathrm{k}} \to \mathrm{Sp}(M) = \mathrm{Sp}_{2m} \to 1$$

  where W is the unique $\mathrm{Sp}(M)$-submod of $\bigwedge^2 M$ of codim 1.
- $\mathcal{P}_{\mathrm{k}}$ does have a Levi factor (over k, not just over $\overline{\mathrm{k}}$)
- but $H^1(\mathrm{Sp}(M), W) \neq 0$ if $m \equiv 0 \pmod{p}$. Distinct classes in this H^1 determine non-conj Levi factors of $\mathcal{P}_{\mathrm{k}}$.

Outline

Sub-systems of a root system

Let Φ an irred root sys in fin dim $\mathbf{Q}$-vector space V with basis Δ.

- For $x \in V$ let $\Phi_x = \{\alpha \in \Phi \mid \langle \alpha, x \rangle \in \mathbf{Z}\}$.
- Φ_x is independent of W_a-orbit of $x \in V$. Thus may suppose that x is in the "basic" alcove A in V, whose walls W_β are labelled by elements β of $\Delta_0 = \Delta \cup \{\alpha_0\}$ where $\alpha_0 = -\widetilde{\alpha}$ is the negative of the "highest root" $\widetilde{\alpha}$.

Proposition

Φ_x is a root subsystem with basis

$$\Delta_x = \{\beta \in \Delta_0 \mid x \in W_\beta\}$$

μ-homomorphisms

For a field F consider the group scheme μ_n which is the kernel of $x \mapsto x^n : \mathbf{G}_m \to \mathbf{G}_m$

Proposition

Let G be a connected linear algebraic group over F. If $\phi : \mu_n \to G$ is a homomorphism, then the image of ϕ is contained in a maximal torus of G.

- The Prop. is something of a *"Folk Theorem"*. I wrote two proofs down in my recent manuscript (one that Serre sketched to me by email in 2007). There are recent proofs in print also by B. Conrad, and by S. Pepin Lehalleur.
- when $p \mid n$, note that μ_n is not a smooth group scheme. Idea behind proof: when $n = p$, homomorphism $\mu_p \to H$ correspond to elements $X \in \mathrm{Lie}(H)$ with $X = X^{[p]}$.

μ-homomorphisms in split tori

- For a linear algebraic group H, view homomorphisms $\phi : \mu_n \to H$ and $\psi : \mu_m \to H$ as equivalent if there is N with $n|N$ and $m|N$ such that

$$\mu_N \to \mu_n \xrightarrow{\phi} H \quad \text{and} \quad \mu_N \to \mu_m \xrightarrow{\psi} H$$

  coincide.
- Equivalence classes are "μ-homomorphisms" – written $\phi : \mu \to H$.

Proposition

If T is a split torus over F with cocharacter group $Y = X_*(T)$, there is a bijection $\overline{x} \mapsto \phi_x$

$$Y \otimes \mathbf{Q}/\mathbf{Z} = V/Y \xrightarrow{\sim} \{\mu\text{-homomorphisms } \mu \to T\}$$

where $V = Y \otimes \mathbf{Q}$.

Some maximal rank subgroups of a reductive group

Let G be a reductive group over F.

Proposition

Let $\phi : \mu \to G$ be a μ-homomorphism.

- *The conn centralizer $M = C^0_G(\phi)$ is a reduc subgp containing a max torus of G; we'll call it a reductive subgroup of type μ.*
- *After extending scalars, ϕ takes values in some split torus T of G. Thus, $\phi = \phi_x$ for some $\overline{x} \in V/Y$ where $Y = X_*(T)$ and $V = Y \otimes \mathbf{Q}$. Then the root system of M is Φ_x.*

Remark

- In char. 0, subgps of type μ have been called "pseudo-Levis".
- Reduc subgps "of type μ" described above account for *some* of the reduc subgps containing a maximal torus. Recipe of Borel and de Siebenthal described all such subgroups.

Examples of subgroups of type μ

"$A_2 \subset G_2$"

- If G is split simple of type G_2, there is a μ-homomorphism $\phi : \mu_3 \to G$ such that $M = C^0_G(\phi) \simeq$ "A_2" $= \mathrm{SL}_3$.
- In char. 3, M is not the connected centralizer of a semisimple element of G.

Outline

Parahoric group schemes, more preciesly

Let G split reduc over K, let T be a split maximal torus, $\Phi \subset X^*(T)$ the roots, and U_α root subgroup for $\alpha \in \Phi$.

- There is a split reduc gp scheme $\mathcal{G}$ over $\mathcal{A}$ with $G = \mathcal{G}_{\mathrm{K}}$.
- Thus, there is a *Chevalley system*: a split $\mathcal{A}$-torus $\mathcal{T}$ and $\mathcal{A}$-forms $\mathcal{U}_\alpha$ for $\alpha \in \Phi$ (plus axioms I'm suppressing).
- A point $x \in V = X_*(T) \otimes \mathbf{Q}$ yields an $\mathcal{A}$-group scheme $\mathcal{U}_{\alpha,x}$ determined from $\mathcal{U}_\alpha$ by the ideal $\pi^m\mathcal{A}$ where $m = \lceil \langle \alpha, x \rangle \rceil$

Theorem (Bruhat and Tits)

The schematic root datum $(\mathcal{T}, \mathcal{U}_{\alpha,x})$ determines a smooth $\mathcal{A}$-group scheme $\mathcal{P} = \mathcal{P}_x$ with $\mathcal{P}_{\mathrm{K}} = G$.

Remark

The $\mathcal{P}_x$ are (up to $G(\mathrm{K})$-conjugacy) the parahoric group schemes attached to G.

Parahoric group schemes, more precisely (continued)

- If G splits over an unramified extension $\mathrm{L} \supset \mathrm{K}$, the parahorics ("over $\mathcal{A}$") arise via étale descent from parahorics for G_{L}.
- To handle general G, Bruhat and Tits also describe parahorics for any quasi-split G.

Main result

Suppose that G splits over an unramified extension of K and let $\mathcal{P}$ be a parahoric group scheme attached to G.

Theorem (McNinch 2018b)

There is a reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ such that:

(a) *$\mathcal{M}_{\mathrm{K}}$ is a reductive subgroup of G of type μ, and*

(b) *$\mathcal{M}_{\mathrm{k}}$ is a Levi factor of the special fiber $\mathcal{P}_{\mathrm{k}}$.*

Remark

- The result is valid for imperfect k.

Sketch of proof

- Via étale descent, may reduce to case G split over K.
- Now $\mathcal{P} = \mathcal{P}_x$ for $x \in V = Y \otimes \mathbf{Q}$. Let $\phi = \phi_x : \mu :\to \mathcal{T}$ be the μ-homomorphism (over $\mathcal{A}$) determined by $\overline{x} \in V/Y$.
- Since $\mu_{N/\mathcal{A}}$ is diagonalizable gp scheme, the centralizer $C_{\mathcal{P}}(\phi)$ is a closed and smooth subgp scheme of $\mathcal{P}$.
- Let $\mathcal{M} = C^0_{\mathcal{P}}(\phi)$ identity component. Then $\mathcal{M}$ is smooth and $\mathcal{M}_{\mathrm{K}}$ has the correct description.
- Must argue $\mathcal{M}$ reductive. Since smooth, only remains to show $\mathcal{M}_{\mathrm{k}}$ reductive.
- That $\mathcal{M}_{\mathrm{k}}$ is a Levi factor of $\mathcal{P}_{\mathrm{k}}$ will follow from fact that Φ_x is the root system of $\mathcal{P}_{\mathrm{k}}/R$.

Examples

Let $\mathrm{K} \subset \mathrm{L}$ be a ramified cubic galois extension, and let G be a quasi-split K-group of type ${}^3D_4$ splitting over L. Assume the residue char p is $\neq 2$.

- A max torus S containing a max split torus T has the form
$$S = R_{\mathrm{L}/\mathrm{K}}\mathbf{G}_m \times \mathbf{G}_m.$$
- Consider the split $\mathcal{A}$-torus $\mathcal{T}$ underlying T.
- Let $\mathcal{P}$ be the parah determined by $\mathcal{S} = R_{\mathcal{B}/\mathcal{A}}\mathbf{G}_m \times \mathbf{G}_m$ and what Bruhat-Tits call the *Chevalley-Steinberg valuation* of G.
- The reductive quotient of $\mathcal{P}_{\mathrm{k}}$ is split of type G_2.
- $\mathcal{P}$ can't have reductive subgroup scheme $\mathcal{M}$ of the form described in the main Theorem.

Outline

Application to nilpotent orbits

Theorem (McNinch 2008, McNinch 2018a)

Let $\mathcal{G}$ be a reductive group scheme over $\mathcal{A}$, and assume that the fibers of $\mathcal{G}$ are standard reductive groups. If $X \in \mathrm{Lie}(\mathcal{G}_{\mathrm{k}})$ is nilpotent, there is a section $\mathcal{X} \in \mathrm{Lie}(\mathcal{G})$ such that

- *$\mathcal{X}_{\mathrm{K}}$ is nilpotent, $\mathcal{X}_{\mathrm{k}} = X$*
- *the centralizers $C_{\mathcal{G}_{\mathrm{k}}}(\mathcal{X}_{\mathrm{k}})$ and $C_{\mathcal{G}_{\mathrm{K}}}(\mathcal{X}_{\mathrm{K}})$ are smooth of the same dimension.*

We say that the nilpotent section $\mathcal{X}$ is a *balanced* nilpotent section lifting X.

Application to nilpotent orbits (continued)

Now let G be reductive over K, suppose that G splits over unramif. ext, and let $\mathcal{P}$ be a parahoric for G. Choose reductive subgroup scheme $\mathcal{M} \subset \mathcal{P}$ as in the *main theorem*. Suppose that $p = \text{char}(\mathrm{k}) > 2h - 2$ where h is the sup of the Coxeter numbers of simple compenents of $G_{\overline{\mathrm{K}}}$.

Theorem (McNinch 2018a)

Let $X \in \text{Lie}(\mathcal{P}_{\mathrm{k}}/R_u\mathcal{P}_{\mathrm{k}}) = \text{Lie}(\mathcal{M}_{\mathrm{k}})$ be nilpotent, and choose $\mathcal{X} \in \text{Lie}(\mathcal{M})$ a balanced nilpotent section for $\mathcal{M}$ lifting X. Then $\mathcal{X}$ is balanced for $\mathcal{P}$ – i.e. the centralizers $C_{\mathcal{P}_{\mathrm{k}}}(\mathcal{X}_{\mathrm{k}})$ and $C_{\mathcal{P}_{\mathrm{K}}}(\mathcal{X}_{\mathrm{K}})$ are smooth of the same dimension.

Application to nilpotent orbits (conclusion)

The assignment $X \mapsto \mathcal{X}_{\mathrm{K}}$ gives another point of view on DeBacker's description (DeBacker 2002) of $G(\mathrm{K})$-orbits on nipotent elements of Lie(G).

In particular:

- Every $X_0 \in \text{Lie}(G)$ has the form $\mathcal{X}_{\mathrm{K}}$ for some balanced section $\mathcal{X} \in \text{Lie}(\mathcal{P})$ for some parahoric $\mathcal{P}$ attached to G.
- For $X_0 \in \text{Lie}(G)$ the ramification behavior of tori in $C_G^0(X)$ constrains the possible $\mathcal{P}$ for which there is $\mathcal{X} \in \text{Lie}(\mathcal{P})$ with $X_0 = \mathcal{X}_{\mathrm{K}}$.

Bibliography

- DeBacker, Stephen (2002). "Parametrizing nilpotent orbits via Bruhat-Tits theory". In: *Ann. of Math. (2)* 156.1, pp. 295–332.
- McNinch, George (2008). "The centralizer of a nilpotent section". In: *Nagoya Math. J.* 190, pp. 129–181.
- – (2010). "Levi decompositions of a linear algebraic group". In: *Transform. Groups* 15.4, pp. 937–964.
- – (2014). "Levi factors of the special fiber of a parahoric group scheme and tame ramification". In: *Algebr. Represent. Theory* 17.2, pp. 469–479.
- – (2018a). *On the nilpotent orbits of a reductive group over a local field*. preprint. URL: http://math.tufts.edu/faculty/gmcninch/manuscripts.html.
- – (2018b). "Reductive subgroup schemes of a parahoric group scheme". In: *Transf. Groups* to appear.