18.024–ESG Exam 2

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1. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by the equations below. Compute the total derivatives $D\mathbf{f}$, $D\mathbf{g}$, and $D(\mathbf{f} \circ \mathbf{g})$.

$$\begin{aligned} \mathbf{f}(x, y) &= (e^x, x + y) \\ \mathbf{g}(u, v, w) &= (uv, uw) \end{aligned}$$

2. The following equation defines z implicitly as a function of x and y. Compute $\partial z/\partial x$ and $\partial z/\partial y$.

$$x^2 z - xz^2 = 2y$$

Answer four of the five parts in questions 3–6.

3. Decide whether the following statements are true or false.

   (a) Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}^n$ be a function given by $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t)) = f_1(t)\mathbf{e}_1 + \cdots f_n(t)\mathbf{e}_n$. If each of the scalar functions $f_1, \ldots, f_n$ is continuous at $t = 0$, then so is $\mathbf{f}$.

   (b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar field. We can define new functions of one variable by $g_1(t) = f(t, 0, \ldots, 0)$, $g_2(t) = f(0, t, 0, \ldots, 0)$, $\ldots$, $g_n(t) = f(0, \ldots, 0, t)$. If each of $g_1, \ldots, g_n$ is continuous at $t = 0$, then f is continuous at $(0, \ldots, 0)$.

4. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0, 0)$; furthermore, suppose that $D_1 f(0, 0) = 5$ and $D_2 f(0, 0) = -2$. Compute the directional derivative $f'((0, 0); (1, 3))$.

5. Let $f : \mathbb{R}^4 \to \mathbb{R}$ be the scalar field $f(x, y, z, w) = xy^2 - z^2 w$. The equation $f(x, y, z, w) = 1$ defines a 3-dimensional hypersurface in $\mathbb{R}^4$. Find a normal vector to this surface at the point $(1, 3, 2, 2)$.

6. Decide whether the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$ exists. If it exists, compute it; if not, explain why not.

7. (Optional) Many of you have given talks about poorly-behaved functions, such as one with all directional derivatives but no total derivative, or one whose second-order mixed partial derivatives do not agree. What technical term is used in mathematical writing to describe such functions? (*Hint*: The answer is not “functions from New Jersey.”)