18.024–ESG Exam 3

Pramod N. Achar

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1. Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field given by $\mathbf{f}(x, y) = (3x^2 - 4xy)\mathbf{i} + (3y^2 - 5xy)\mathbf{j}$. Let C be the curve from $(-2, 4)$ to $(1, 1)$ along the parabola $y = x^2$.

   (a) Compute the length of this curve. (*Hint*: You may find the following formula useful: $\int \sqrt{1 + u^2} = \frac{1}{2}(u\sqrt{1 + u^2} + \operatorname{arcsinh} u)$.)

   (b) Compute the line integral $\int_C \mathbf{f}$.

   (c) Show that $\mathbf{f}$ is not a gradient.

2. Let $S \subset \mathbb{R}^2$ be a region that is not a rectangle, and let $f : S \to \mathbb{R}$ be some scalar function defined on S. How is the integral $\iint_S f$ defined, in terms of integrals over rectangles?

3. Let S be the region in $\mathbb{R}^2$ bounded by the graphs of $y = x^2$ and $y = x^3$. (These curves intersect at two points—you may want to sketch them to see exactly what S looks like.) Compute $\iint_S (30x^2 - 210y^2)$.

4. Find a potential function for the vector field $\mathbf{f}(x, y) = 3x^2y\mathbf{i} + x^3\mathbf{j}$.

5. Let $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field. Prove that if $\mathbf{f}$ is the gradient of some scalar field ϕ, then $\partial P/\partial y = \partial Q/\partial x$.

6. (Optional) Which of the following is not a technical term in mathematics?

   (a) excellent
   (b) universally Japanese
   (c) flabby
   (d) rough