18.024–ESG Exam 4

Pramod N. Achar

Spring 2000

1. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{f}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$. Let S be the portion of the surface given by $z = 1 - x^2 - y^2$ lying above the xy-plane.

   (a) Compute the vector field $\mathbf{g} = \operatorname{curl} \mathbf{f}$.

   (b) Using a parametrization of S, convert the surface integral $\iint_S \mathbf{g}$ into a double integral over some region in $\mathbb{R}^2$. Set up the limits on this double integral, and simplify the integrand as much as possible, but do not evaluate it.

   (c) Compute $\iint_S \mathbf{g}$ by using Stokes' Theorem to convert it into a line integral.

2. Let S be the region in $\mathbb{R}^2$ bounded by the curves $xy = 1$, $xy = 2$, $y = x$, and $y = 4x$. Let $f : \mathbb{R} \to \mathbb{R}$ be a scalar function of one variable. Using the change of variables $x = \sqrt{u/v}$, $y = \sqrt{uv}$, show that $\iint_S f(xy) = (\ln 2) \int_1^2 f(u)$.

3. Let $\mathbf{f}$ be a vector field in $\mathbb{R}^3$ that is the curl of some other vector field. Let S be a closed surface (*i.e.*, a surface with no boundary, like a sphere), and suppose that S is the boundary of some 3-dimensional region V. Prove that $\iint_S \mathbf{f} = 0$.

4. In $\mathbb{R}^3$, we know how to integrate over domains of dimension 1, 2, or 3: these are line integrals, surface integrals, and triple integrals respectively; and together, they give us a complete picture of integration theory in $\mathbb{R}^3$. The ideas involved can give us part of the picture of integration theory in higher dimensions. What kinds of integrals (*i.e.* over domains of what dimension) can we do in $\mathbb{R}^n$ without introducing any new algebraic constructions (*e.g.* tensors or matrix fields)? Which integration theorems from $\mathbb{R}^3$ carry over to the higher-dimensional setting? Give a brief explanation of your answer.

5. (Facultatif) Ceci n'est pas une question.