18.024–ESG Notes 1

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Spring 2000

The goal of this set of notes is to make sense of the idea of "dimension" of linear subspaces of $\mathbb{R}^n$. An intuitive idea of dimension might say that it is the number of parameters needed to specify a point, or the number of degrees of freedom, in some sense. To make this idea formal, we have introduced the notions of *spanning set* and *linear independence*. Roughly, the idea of a spanning set corresponds to having enough parameters to describe points in the space, and the idea of linear independence corresponds to nonredundancy among those parameters.

Let $W \subset \mathbb{R}^n$ be a linear subspace. A given set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in W$ might span W without being linearly independent (*i.e.*, they might be redundant); or, they might be linearly independent but not span W. (Of course, they might also be neither linearly independent nor a spanning set.) A spanning set that is not linearly independent is too large, and a linearly independent set that does not span is too small. A *basis* for a linear subspace is a linearly independent spanning set. The size of a basis should be "just right" to describe points in the subspace, in the following sense:

- Every vector in the subspace can be written as a linear combination of the basis vectors (because the basis is a spanning set).
- There is only one way of writing a given vector as a linear combination of basis vectors (because they are linearly independent).

With these observations in hand, we can make the following definition:

Definition 1. The *dimension* of a linear subspace is the number of vectors in a basis for that subspace.

But wait! What if different bases have different sizes? Which one determines the dimension? Fortunately, this potential problem does not actually occur, but before we can start flippantly bandying about the word *dimension*, we need to prove that that problem does not occur.

Lemma 2. *Let $W \subset \mathbb{R}^n$ be a linear subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Then any set of $k+1$ vectors in W is linearly dependent.*

Proof. We prove this lemma by induction, so let us begin with the case $k = 1$. That is to say, W is spanned by just one vector, $\mathbf{v}_1$. We want to show that if $\mathbf{u}_1$ and $\mathbf{u}_2$ are any two vectors in W, then they are linearly dependent. Because $\mathbf{u}_1$ and $\mathbf{u}_2$ are contained in W, they can be written in terms of the spanning set we have for W, *viz.*, the single vector $\mathbf{v}_1$:

$$\begin{aligned} \mathbf{u}_1 &= c_1\mathbf{v}_1 \\ \mathbf{u}_2 &= c_2\mathbf{v}_1 \end{aligned} \tag{1}$$

To show that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent, we need to exhibit a nontrivial way of writing $\mathbf{0}$ as a linear combination of them. Let us break it down into two cases (I did not do this explicitly in class).

Case 1. Both c_1 and c_2 are zero. This means $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$, so any linear combination of them whatsoever will give $\mathbf{0}$. For example, $5\mathbf{u_1} + 3\mathbf{u_2} = \mathbf{0}$.

Case 2. The coefficients c_1 and c_2 are not both zero. From Equation (1), we can derive

$$c_2\mathbf{u}_1 - c_1\mathbf{u_2} = \mathbf{0}. \tag{2}$$

Since c_1 and c_2 are not both zero, this equation shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent, as desired.

For the next part of the inductive proof, we assume the lemma to be true for $k = m$, and prove it for $k = m + 1$. To reiterate our assumption, we suppose that in *any* subspace spanned by m vectors, *any* set of $m + 1$ (or more) vectors is linearly dependent. We want now to show that if W is a space spanned by, say, $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$, then any set of $m + 2$ vectors in W, say $\mathbf{u}_1, \ldots, \mathbf{u}_{m+2}$, must be linearly dependent. Since the $\mathbf{u}_i$'s are in W, they can be written in terms of the $\mathbf{v}_i$'s.

$$\begin{aligned}
\mathbf{u}_1 &= \quad c_{1,1}\mathbf{v}_1 + \quad c_{1,2}\mathbf{v}_2 + \cdots + \quad c_{1,m+1}\mathbf{v}_{m+1} \\
\mathbf{u}_2 &= \quad c_{2,1}\mathbf{v}_1 + \quad c_{2,2}\mathbf{v}_2 + \cdots + \quad c_{2,m+1}\mathbf{v}_{m+1} \\
&\vdots \\
\mathbf{u}_{m+2} &= c_{m+2,1}\mathbf{v}_1 + c_{m+1,2}\mathbf{v}_2 + \cdots + c_{m+2,m+1}\mathbf{v}_{m+1}
\end{aligned}$$

We split the remainder of the argument into two cases, just as we did for $k = 1$.

Case 1. Suppose that the coefficients in the last column, $c_{1,m+1}, \ldots, c_{m+2,m+1}$, are all zero. Let us define the new subspace W' to be the linear span of $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Since each of $\mathbf{u}_1, \ldots, \mathbf{u}_{m+2}$ has a zero coefficient for $\mathbf{v}_{m+1}$, it actually lies in the smaller subspace W'. The $\mathbf{u}_i$'s are a set of $m + 2$ vectors sitting in a subspace W' that is spanned by just m vectors, so the inductive assumption tells us that they must be linearly dependent.

Case 2. On the other hand, we need to deal with the case that some of the $c_{i,m+1}$'s are nonzero. Let us suppose, in particular, that the last coefficient $c_{m+2,m+1}$ is nonzero. (If some other coefficient were nonzero, the proof would be the same, but the definition of the $\mathbf{w}_i$'s below would be changed slightly to use that coefficient instead.) We form combinations of the $\mathbf{u}_i$'s analagous to that in (2), as follows:

$$\begin{aligned}
\mathbf{w}_1 &= c_{m+2,m+1}\mathbf{u}_1 \quad - \quad c_{1,m+1}\mathbf{u}_{m+2} \\
\mathbf{w}_2 &= c_{m+2,m+1}\mathbf{u}_2 \quad - \quad c_{2,m+1}\mathbf{u}_{m+2} \\
&\vdots \\
\mathbf{w}_{m+1} &= c_{m+2,m+1}\mathbf{u}_{m+1} - c_{m+1,m+1}\mathbf{u}_{m+2}
\end{aligned}$$

These combinations are designed so as to have the coefficients of $\mathbf{v}_{m+1}$ cancel out, just as the coefficients of $\mathbf{v}_1$ cancelled in (2). That is, when the above expressions for the $\mathbf{w}_i$'s are exapnded out in terms of the $\mathbf{v}_i$'s, we would see that each $\mathbf{w}_i$ depends only on $\mathbf{v}_1, \ldots, \mathbf{v}_m$, and not on $\mathbf{v}_{m+1}$. Thus the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_{m+1}$ are contained in the subspace W' spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Our inductive assumption tells us that any set of $m + 1$ vectors in W' must be linearly dependent, so there is some equation

$$d_1\mathbf{w}_1 + \cdots + d_{m+1}\mathbf{w}_{m+1} = \mathbf{0},$$

where not all of the d_i's are zero. If we substitute the definitions of the $\mathbf{w}_i$'s in this equation, we get

$$\begin{aligned}
d_1 c_{m+2,m+1}\mathbf{u}_1 + d_2 c_{m+2,m+1}\mathbf{u}_2 + \cdots + d_{m+1}c_{m+2,m+1}\mathbf{u}_{m+1}& \\
- (d_1 c_{1,m+1} + d_2 c_{2,m+1} + \cdots + d_{m+1}c_{m+1,m+1})\mathbf{u}_{m+2} &= \mathbf{0}.
\end{aligned}$$

Since not all the d_i's are zero, and since in particular $c_{m+2.m+1} \neq 0$, this equation must contain some nonzero coefficients. From this equation, we conclude that $\mathbf{u}_1, \ldots, \mathbf{u}_{m+2}$ are linearly dependent, as desired.

Thus, once we have a spanning set, any larger set of vectors is automatically linearly dependent. □

Theorem 3. *Let $W \subset \mathbb{R}^n$ be a linear subspace. Every basis of W contains the same number of a vectors.*

Proof. If we have two spanning sets for W of different sizes, the preceding lemma tells us that the bigger one must necessarily be linearly dependent. So if we have two spanning sets that are also both linearly independent, then they must contain the same number of vectors. □