In this set of notes, we will develop the basic theory of tensors and differential forms; we will learn what it means to integrate a differential form; and we will state the generalized Stokes' Theorem in terms of differential forms. Caveat lector: there are several closely related meanings of the word tensor.

We begin introducing some convenient terminology for which we will not give precise definitions. A manifold is a curve, surface, or higher-dimensional generalization thereof. We will often shorten the phrase “manifold of dimension $k$” to “$k$-manifold.” When we integrate over a $k$-manifold $M$, we will need to make use of a parametrization $r: D \to M$, where $D \subset \mathbb{R}^k$. If $M \subset \mathbb{R}^n$, then we say the codimension of $M$ is $n-k$. Thus, curves are 1-manifolds. 2-manifolds sitting in $\mathbb{R}^3$ have codimension 1, but if they sit in $\mathbb{R}^4$, they have codimension 2. Note that it makes sense to speak of “normal vectors” to manifolds of codimension 1.

In this set of notes, higher-dimensional integrals will never be written as multiple integrals.

Definition 1. A tensor of degree $k$, or a $k$-tensor, on $\mathbb{R}^n$ is a real-valued function of $k$ variables, where each input variable is a vector in $\mathbb{R}^n$, such that the function is linear in each variable. That is, a tensor is a function

$$T: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R},$$

such that

$$T(v_1, \ldots, cv_i + c'v'_i, \ldots, v_k) = cT(v_1, \ldots, v_i, \ldots, v_k) + c'T(v_1, \ldots, v'_1, \ldots, v_k)$$

for each $i$. By convention, a 0-tensor is a just a real constant.

The first thing to notice about tensors is that because they must be linear in each variable, then if we specify the values of a tensor on the unit coordinate vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$, then that determines the tensor uniquely.

What does this mean? Let us consider the simplest nontrivial case, that of a 1-tensor. This is just a linear function $T: \mathbb{R}^n \to \mathbb{R}$. If we know, for example, that $T(e_1) = a_1, \ldots, T(e_n) = a_n$, then we can compute $T(v)$ for any vector $v \in \mathbb{R}^n$. Suppose that

$$v = v_1e_1 + \cdots + v_ne_n;$$

then, by linearity of $T$, we must have

$$T(v) = v_1a_1 + \cdots + v_na_n.$$ 

This expression can be written more concisely: if we let $a \in \mathbb{R}^n$ be the vector $a_1e_1 + \cdots + a_ne_n$, then

$$T(v) = a \cdot v.$$ 

This discussion illustrates the relationship between vectors and tensors of degree 1. Indeed, the ideas in this paragraph could be made into a rigorous proof of the following fact.
Proposition 2. There is a one-to-one correspondence between vectors in \( \mathbb{R}^n \) and tensors of degree 1 on \( \mathbb{R}^n \). The correspondence is as follows: if \( T : \mathbb{R}^n \to \mathbb{R} \) is a tensor, the corresponding vector is

\[
(T(e_1), \ldots, T(e_n));
\]

conversely, given \( a \in \mathbb{R}^n \), the corresponding tensor is defined by

\[
T(v) = a \cdot v.
\]

Some people, such as physicists, do not distinguish between a vector and a 1-tensor. We say that a tensor is determined by its values on the unit coordinate vectors. If \( T \) is a function of two vector variables, we need to evaluate it on all pairs of unit coordinate vectors. Let

\[
a_{ij} = T(e_i, e_j);
\]

there are \( n^2 \) such numbers. As we might expect, 2-tensors are closely related to \( n \times n \) matrices, but the algebra is more complicated than in the degree 1 case. The precise relationship is given in the following proposition, which we state without proof.

Proposition 3. There is a one-to-one correspondence between \( n \times n \) matrices and tensors of degree 2 on \( \mathbb{R}^n \). The correspondence is as follows: if \( T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is a tensor, the corresponding matrix is

\[
\begin{pmatrix}
T(e_1, e_1) & T(e_1, e_2) & \cdots & T(e_1, e_n) \\
T(e_2, e_1) & T(e_2, e_2) & \cdots & T(e_2, e_n) \\
\vdots & \vdots & \ddots & \vdots \\
T(e_n, e_1) & T(e_n, e_2) & \cdots & T(e_n, e_n)
\end{pmatrix};
\]

conversely, given an \( n \times n \) matrix \( A \), the corresponding tensor is defined by

\[
T(v, w) = v \cdot A w.
\]

Again, physicists often think of the array of scalars \( a_{ij} \) as being the tensor, whereas mathematicians think of it as determining a tensor.

It is difficult to continue this analogy to tensors of degree greater than 2, but we can still make the observation that a tensor is determined by its values on the unit coordinate vectors. If \( T \) is a \( k \)-tensor on \( \mathbb{R}^n \), consider the constants

\[
a_{i_1i_2\ldots i_k} = T(e_{i_1}, e_{i_2}, \ldots, e_{i_k}).
\]

There are \( n \) choices for which unit coordinate vector to use in the first variable, \( n \) choices for the second, and so on, for a total of \( n^k \) constants \( a_{i_1i_2\ldots i_k} \). These \( n^k \) constants determine the tensor. It is possible to imagine these \( n^k \) constants arranged in a \( k \)-dimensional array with \( n \) rows in each direction, but when \( k > 2 \), this is not necessarily a useful way to think of tensors.

Given any two tensors \( S \) and \( T \), there is an easy way to form a new tensor of higher degree out of them.

Definition 4. Let \( S \) be a \( k \)-tensor on \( \mathbb{R}^n \), and \( T \) an \( l \)-tensor. Their tensor product \( S \otimes T \) is a new tensor of degree \( k + l \), defined by

\[
(S \otimes T)(v_1, \ldots, v_{k+l}) = S(v_1, \ldots, v_k)T(v_{k+1}, \ldots, v_{k+l}).
\]

Definition 5. An alternating tensor of degree \( k \) on \( \mathbb{R}^n \) is a \( k \)-tensor with the property that exchanging any two of its input variables negates its value. That is, \( T \) is alternating if

\[
T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -T(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).
\]

This condition is vacuously satisfied by all tensors of degree 0 and 1, so we say that all degree 0 or 1 tensors are alternating. The set of all alternating \( k \)-tensors on \( \mathbb{R}^n \) is denoted by \( \Lambda^k(\mathbb{R}^n)^* \).
It turns out that in the correspondence between 2-tensors and matrices, alternating tensors correspond precisely to skew-symmetric matrices. (This is why, in Problem Set 12, you integrated a skew-symmetric matrix field over a 2-manifold in $\mathbb{R}^4$—you were really integrating an alternating 2-tensor.)

We are now almost ready to give a general definition of integration.

**Definition 6.** Let $A \subset \mathbb{R}^n$. A *differential form* of degree $k$ on $A$, also called a $k$-form or an *alternating $k$-tensor field*, is a function $f : A \rightarrow \Lambda^k(\mathbb{R}^n)^*$.

This last definition can be a little hard to get your head around: remember that the elements of $\Lambda^k(\mathbb{R}^n)^*$ are tensors, which are themselves functions. So a differential form is a function that assigns to each point of its domain another function! This is a little less mind-boggling if we instead think of a form as assigning an array of $n^k$ constants to each point, but it is important to remember that if $\omega$ is a $k$-form, and $x$ is a point of its domain, then $\omega(x)$ is itself a function of $k$ vector variables.

We take as given the definition and properties of $k$-dimensional integrals of scalar functions over regions of $\mathbb{R}^k$: i.e., ordinary multiple integrals. We now define integration of differential forms over manifolds in terms of ordinary multiple integrals.

**Definition 7.** Let $M$ be a $k$-manifold in $\mathbb{R}^n$, and let $\gamma : D \rightarrow M$ be a parametrization of it, where $D \subset \mathbb{R}^k$. Let $\omega$ be a $k$-form defined on some subset of $\mathbb{R}^n$ containing $M$. Let us write $t_1, \ldots, t_k$ for the variables in $\mathbb{R}^k$. Then, we define

$$\int_M \omega = \int_D \omega \left( \frac{\partial \gamma}{\partial t_1}, \ldots, \frac{\partial \gamma}{\partial t_k} \right).$$

As always, there is the question of whether this is well-defined; i.e., whether the value of $\int_M \omega$ depends on the parametrization of $M$ that we choose. This is why we need to require that $\omega$ be an alternating tensor field and not an arbitrary one. Here is the desired fact, without proof:

**Theorem 8.** The integral of a $k$-form over a $k$-manifold is well-defined up to sign, independent of the parametrization.

This fits well with what we have seen before for line integrals and surface integrals: a line integral changes sign if the curve is parametrized in the opposite direction, and a surface integral changes sign if the opposite normal direction is used. But in all these cases, the value of the integral is well-defined up to a sign.

The last major thing we need to do is to produce an analogue of the Second Fundamental Theorem. Suppose that $M$ is a $k$-manifold with boundary $N$, a $(k-1)$-manifold; and let $\omega$ be a $(k-1)$-form. Following the pattern of the Second Fundamental Theorem, Green’s Theorem, Stokes’ Theorem, etc., we expect a theorem that tells us that $\int_N \omega$ is equal to the integral of some related form over $M$, where that related form should be obtained by doing some sort of differentiation on $\omega$. It should also be a form of degree one greater than that of $\omega$, since $M$ is a manifold of dimension one greater than that of $N$.

This is where we introduce the differential operator $\partial$. It has a uniform definition for all forms of all degrees in all dimensions, instead of idiosyncratic definitions like those we have seen for curl, divergence, etc. We will define it first for 0-forms, and then say how to compute the differential of a form of higher degree in terms of differentials of lower-degree forms. To achieve this reduction, we need a way to get higher-degree forms from lower-degree ones. The problem is that the tensor product operation $\otimes$ does not preserve the property of being alternating. That is, even if $S$ and $T$ are both alternating tensors, $S \otimes T$ almost never is.

Before we can fix this problem, we need to investigate in much greater detail what it means for a tensor to be alternating. An arbitrary $k$-tensor is determined by $n^k$ constants, but most tensors are not alternating, so it should take a lot fewer constants to determine an alternating tensor. We have already seen this with $4 \times 4$ skew-symmetric matrices: it takes 16 numbers to determine a 2-tensor on $\mathbb{R}^4$, but only 6 numbers to determine an alternating 2-tensor on $\mathbb{R}^4$. (There are 6 numbers to specify in giving a $4 \times 4$ skew-symmetric matrix.)

How does this come about? To say $T$ is alternating is to say that exchanging any two input variables changes the sign of the answer. It follows that if any two input variables to $T$ are equal, the output value must be 0 (since it must equal its own negative). Thus, among the constants

$$a_{i_1i_2\ldots i_k} = T(e_{i_1}, e_{i_2}, \ldots, e_{i_k}),$$
we get 0 whenever two of the indices $i_1, \ldots, i_k$ are equal. (Hence the 0's along the diagonal of a skew-symmetric matrix.) The constants can be nonzero only if all $k$ indices in the subscript are distinct. Moreover, the alternating property means that

$$a_{i_1 \ldots i_p \ldots i_q \ldots i_k} = -a_{i_1 \ldots i_q \ldots i_p \ldots i_k}.$$ 

Any rearrangement of the indices can be achieved by a series of exchanges like this, so there is really only one parameter to be determined among all the possible rearrangements of $i_1 \ldots i_k$. (Thus there 12 nonzero entries in a skew-symmetric matrix, but 6 of them are the negatives of the other 6, so there are really only 6 parameters to be determined.) Hence, the number of parameters needed to determine an alternating $k$-tensor is equal to the number of ways of choosing $k$ distinct vectors from among the $n$ unit coordinate vectors $e_1, \ldots, e_n$. This is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$ 

And, indeed, it takes $(\frac{4}{2}) = 6$ parameters to determine an alternating 2-tensor on $\mathbb{R}^4$. One consequence of this is that, on $\mathbb{R}^n$, there are no alternating tensors of degree greater than $n$ (except the zero tensor).

In any case, let us return to examining the constants $a_{i_1 \ldots i_k}$. Recall that a *permutation* $\sigma$ of a set $k$ objects is just a rearrangement of them. If $\sigma$ is any such permutation, we have

$$a_{i_1 \ldots i_k} = \pm a_{\sigma(i_1 \ldots i_k)}.$$ 

Any permutation can be performed by doing a sequence of pairwise exchanges; the sign on the left-hand side of the above equation depends on whether the number of such exchanges required is odd or even.

**Definition 9.** The *sign* of a permutation $\sigma$, denoted by $\text{sgn} \sigma$, is either $+1$ or $-1$, whichever appears as the coefficient on the left-hand side of the above equation. A permutation with sign $+1$ is called even, and one with sign $-1$ is called odd.

An important point to note here is that the sign of a permutation is an intrinsic property of the permutation, having nothing to do with tensors or forms.

**Definition 10.** Let $S$ be an alternating $k$-tensor on $\mathbb{R}^n$, and $T$ and alternating $l$-tensor. Their *exterior* or *wedge product* $S \wedge T$ is a new alternating tensor of degree $k+l$, defined by

$$(S \wedge T)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\text{all permutations } \sigma \text{ of } k+l \text{ objects}} (\text{sgn } \sigma)(S \otimes T)(\sigma(v_1, \ldots, v_{k+l})).$$