18.024–ESG Problem Set 1

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18.014 Review Questions

Give answers only, no proofs, to the following questions. Try to answer as many as you can without consulting Apostol or your notes.

1. Which of the sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields? Which are ordered fields?

2. Determine whether the following two statements are true or false.

   (a) If f is integrable on $[a, b]$, then it is continuous on that interval.

   (b) If f is continuous on $[a, b]$, then it is integrable on that interval.

3. Bolzano's Theorem is related to the Intermediate-Value Theorem in the same way that Rolle's Theorem is related to the Mean-Value Theorem. Briefly describe this relationship. Try to give an intuitive explanation rather than an overly technical one.

4. What theorem did you use to prove Brouwer's Fixed-Point Theorem? (Well, the 1-dimensional version—you will be able to prove Brouwer's Fixed-Point Theorem in higher dimensions after taking 18.905.)

5. Let f be an integrable function, and define a new function F by the formula $F(x) = \int_a^x f$. What condition must f satisfy in order to guarantee that F is differentiable? In that case, what is the derivative of F?

6. Is the Intermediate-Value Theorem still true for functions whose domain is the disjoint union of two closed intervals (*i.e.*, something like $[0, 1] \cup [2, 3]$)? What about the Extreme-Value Theorem?

7. To what other theorem is the zeroth-order version of Taylor's Theorem equivalent?

8. What does it mean for a sequence $a : \mathbb{Z}_+ \to \mathbb{R}$ to converge? Give a rigorous definition.

9. Suppose that $\sum u_n(x)$ is a series of functions converging pointwise to $F(x)$. Does $\sum Du_n$ necessarily converge to DF? Does $\sum \int u_n$ necessarily converge to $\int F$? Answer the same two questions under the additional assumption that the convergence of $\sum u_n(x)$ is uniform.

Tuesday

10. Exercises 3 and 4 in Section 12.8 of Apostol, Volume I. (*N.B.*: Apostol uses the capital letter "O" to denote the zero vector, which I denote by $\mathbf{0}$ or $\vec{0}$.)

Thursday

11. Exercise 1 in Section 12.11 of Apostol, Volume I.

12. Let θ be the angle between two nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Prove the "law of cosines":

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta.$$

Friday

13. Exercises 3 and 4 in Section 12.15 of Apostol, Volume I.

14. Exercise 17 in Section 12.15 of Apostol, Volume I.