18.024–ESG Problem Set 12

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In this problem set you will prove an analogue of the Fundamental Theorem of Calculus, Stokes' Theorem, *etc.* To set up the context of the problem, let us define an operator div° on vector fields on $\mathbb{R}^2$ by

$$\operatorname{div}^\circ(P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Let C denote a curve in $\mathbb{R}^n$ with endpoints $\mathbf{a}$ and $\mathbf{b}$. Let S denote a surface whose boundary is the closed curve $\operatorname{Bd} S$. Let V denote a solid region in $\mathbb{R}^3$ with boundary $\operatorname{Bd} V$. The following table summarizes the state of our knowledge of integration theory in three or fewer dimensions.

$\mathbb{R}$	$\mathbb{R}^2$	$\mathbb{R}^3$
scalar fields $\big\downarrow D$ scalar fields	scalar fields $\big\downarrow \text{grad}$ vector fields $\big\downarrow \text{div}^\circ$ scalar fields	scalar fields $\big\downarrow \text{grad}$ vector fields $\big\downarrow \text{curl}$ vector fields $\big\downarrow \text{div}$ scalar fields
$\int_a^b Df = f(b) - f(a)$	$\int_C \operatorname{grad}\mathbf{f} = \mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a})$ $\iint_S \operatorname{div}^\circ \mathbf{f} = \int_{\operatorname{Bd} S} \mathbf{f}$	$\int_C \operatorname{grad}\mathbf{f} = \mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a})$ $\iint_S \operatorname{curl}\mathbf{f} = \int_{\operatorname{Bd} S} \mathbf{f}$ $\iiint_V \operatorname{div}\mathbf{f} = \iint_{\operatorname{Bd} V} \mathbf{f}$
	$\operatorname{div}^\circ \operatorname{grad}\mathbf{f} = \mathbf{0}$	$\operatorname{curl}\operatorname{grad}\mathbf{f} = \mathbf{0}$ $\operatorname{div}\operatorname{curl}\mathbf{f} = \mathbf{0}$

There is a theme running through all these theorems: roughly, the above table can be summarized by the following two statements:

(a) An integral of an appropriately "differentiated" function over a region R can be evaluated by an integral of one lower dimension: specifically, the integral of the original function over the boundary of R.

(b) "Appropriately differentiating" a function that has already been appropriately differentiated function always gives $\mathbf{0}$.

Here, the meaning of "appropriately differentiated" depends on context. We expect that the analogous theory in $\mathbb{R}^4$ will have four kinds of "appropriate differentiation" operators, four theorems that reduce the dimension of an integral, and three theorems about repeated differentiation giving zero.

An additional leap is required in the setting of $\mathbb{R}^4$: we need to introduce *matrix-valued functions* (or *matrix fields*), and we shall need to define integration of matrix fields over 2-dimensional surfaces in $\mathbb{R}^4$.

Skew-Symmetric Matrices and Differentiation in $\mathbb{R}^4$

A *skew-symmetric matrix* is a square matrix equal to the negative of its transpose. In other words, if a_{ij} denotes the entry in row i, column j of a matrix A, then we say that A is *skew-symmetric* if $a_{ij} = -a_{ji}$ for every i, j. Let $S(n)$ denote the set of all $n \times n$ skew-symmetric matrices.

For example, $\left(\begin{smallmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{smallmatrix}\right)$ is a 3×3 skew-symmetric matrix. Note that the entries on the diagonal must be 0, since they are required to be their own negatives.

If $\mathbf{f} : \mathbb{R}^4 \to \mathbb{R}^4$ is a vector field, say

$$\mathbf{f}(x, y, z, w) = P\,\mathbf{e}_1 + Q\,\mathbf{e}_2 + R\,\mathbf{e}_3 + S\,\mathbf{e}_4,$$

then we define a new skew-symmetric matrix field, called $\operatorname{twist} \mathbf{f} : \mathbb{R}^4 \to S(4)$, by

$$\operatorname{twist} \mathbf{f} = \begin{pmatrix} 0 & \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} & \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} & \frac{\partial S}{\partial x} - \frac{\partial P}{\partial w} \\ \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} & 0 & \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} & \frac{\partial S}{\partial y} - \frac{\partial Q}{\partial w} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} & \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} & 0 & \frac{\partial S}{\partial z} - \frac{\partial R}{\partial w} \\ \frac{\partial P}{\partial w} - \frac{\partial S}{\partial x} & \frac{\partial Q}{\partial w} - \frac{\partial S}{\partial y} & \frac{\partial R}{\partial w} - \frac{\partial S}{\partial z} & 0 \end{pmatrix}$$

Next, if $\mathbf{g} : \mathbb{R}^4 \to S(4)$ is a skew-symmetric matrix field, say

$$\mathbf{g}(x, y, z, w) = \begin{pmatrix} 0 & A & B & C \\ -A & 0 & D & E \\ -B & -D & 0 & F \\ -C & -E & -F & 0 \end{pmatrix},$$

then we define a new vector field, called $\operatorname{spin} \mathbf{g} : \mathbb{R}^4 \to \mathbb{R}^4$, by the formula

$$\begin{aligned} \operatorname{spin} \mathbf{g} = &\left(+\frac{\partial F}{\partial y} - \frac{\partial E}{\partial z} + \frac{\partial D}{\partial w}\right)\mathbf{e}_1 + \left(-\frac{\partial F}{\partial x} + \frac{\partial C}{\partial z} - \frac{\partial B}{\partial w}\right)\mathbf{e}_2 \\ &+ \left(+\frac{\partial E}{\partial x} - \frac{\partial C}{\partial y} + \frac{\partial A}{\partial w}\right)\mathbf{e}_3 + \left(-\frac{\partial D}{\partial x} + \frac{\partial B}{\partial y} - \frac{\partial A}{\partial z}\right)\mathbf{e}_4. \end{aligned}$$

In addition to twist and spin, we also have the grad and div operators, defined exactly as they are in $\mathbb{R}^3$.

Integration in $\mathbb{R}^4$

Now we need to discuss integration in $\mathbb{R}^4$. Line integrals are easy: we have dealt with integrating vector fields along curves in full generality, and it works the same way in $\mathbb{R}^4$ as in lower dimensions. If C is a curve and $\mathbf{f}$ is a vector field, then $\int_C \mathbf{f}$ is defined by

$$\int_C \mathbf{f} = \int_a^b \mathbf{f}(\mathbf{r}(t)) \cdot \mathbf{r}'(t),$$

where $\mathbf{r} : [a, b] \to \mathbb{R}^4$ is a parametrization of the curve.

Two-dimensional integrals in $\mathbb{R}^4$ are in some sense the hardest. In this setting the most natural type of function to integrate turns out to be neither scalar fields nor vector fields, but rather skew-symmetric matrix fields. Why is this? In lower dimensions, whenever we integrate a vector field, we convert it into a scalar function by taking the dot product with a tangent or normal vector gotten from a parametrization, and then integrate that scalar field over the parametrizing region by an ordinary multiple integral. But for two-dimensional integrals in $\mathbb{R}^4$, there is no single tangent or normal direction to dot with. Indeed, at every point on the surface there is a whole two-dimensional plane of tangent vectors, and another two-dimensional plane of normal vectors. So the kind of function we integrate over a two-dimensional surface in $\mathbb{R}^4$ should be something that can give us a scalar when we combine it with *two* vectors, not just dot it with one. Skew-symmetric matrices fit the bill.

If $\mathbf{q}$ is a vector in $\mathbb{R}^4$ and $\mathbf{A}$ is a 4×4 skew-symmetric matrix, then $\mathbf{Aq}$ is another vector. (Remember that vectors ought to be regarded as column matrices; *i.e.*, $\mathbf{v}$ can be thought of as a 4×1 matrix. Then the matrix multiplication $\mathbf{Av}$ makes sense, and the answer is another 4×1 matrix—in other words, a vector.) If $\mathbf{p}$ is another vector in $\mathbb{R}^4$, then we can form the dot product $\mathbf{p} \cdot \mathbf{Aq}$, and get a scalar. (Actually, the operation $\mathbf{p} \cdot \mathbf{Aq}$ makes sense for any 4×4 matrix, not just skew-symmetric ones, but it turns out that the definition of integration we are about to give is not independent of the parametrization unless the matrix field is skew-symmetric.)

This is exactly what we do with skew-symmetric matrix fields over two-dimensional surfaces. Let $\mathbf{r} : D \to \mathbb{R}^4$ be a parametrization of a surface S, where $D \subset \mathbb{R}^2$, and let us use the letters s and t for the coordinates in $\mathbb{R}^2$. Each partial derivative $\partial\mathbf{r}/\partial s$ and $\partial\mathbf{r}/\partial t$ is a function $D \to \mathbb{R}^4$; and each of these gives a tangent vector to the surface parametrized by $\mathbf{r}$. If $\mathbf{F}$ is a skew-symmetric matrix field, we define $\iint_S \mathbf{F}$ by

$$\iint_S \mathbf{F} = \iint_D \frac{\partial\mathbf{r}}{\partial s} \cdot \mathbf{F}(\mathbf{r}(s,t)) \frac{\partial\mathbf{r}}{\partial t}.$$

Before we tackle three-dimensional integrals, it will be helpful to recall the

$\mathbb{R}^4$ analogue of cross product: the vector given by the mnemonic formula

$$\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ \longleftarrow & & \mathbf{a} & \longrightarrow \\ \longleftarrow & & \mathbf{b} & \longrightarrow \\ \longleftarrow & & \mathbf{c} & \longrightarrow \end{pmatrix}$$

is perpendicular to each of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Let us introduce the notation

$$\Xi(\mathbf{a}, \mathbf{b}, \mathbf{c})$$

for this vector.

Three-dimensional integrals in $\mathbb{R}^4$ are very similar to two-dimensional (surface) integrals in $\mathbb{R}^3$. In $\mathbb{R}^3$, to integrate a vector field over a surface, we take its dot product with a normal vector to the surface, and then integrate the resulting scalar over the subset of $\mathbb{R}^2$ that parametrizes the surface. To integrate a vector field over a three-dimensional surface in $\mathbb{R}^4$, we do the same thing: take its dot product with a normal vector, and triple-integrate the resulting scalar function over the subset of $\mathbb{R}^3$ that parametrizes the surface. How do we find such a normal vector? For two-dimensional integrals in $\mathbb{R}^3$, we got it as the cross product of the partial derivatives of the parametrizing function. For three-dimensional integrals in $\mathbb{R}^4$, the parametrizing function has three partial derivatives, and we use "Ξ" instead of cross product. Let $S \subset \mathbb{R}^4$ be a three-dimensional surface parametrized by $\mathbf{r} : D \to \mathbb{R}^4$, where $D \subset \mathbb{R}^3$, and let s, t, and u be the coordinates on $\mathbb{R}^3$. Then, the integral of a vector field $\mathbf{f}$ over S is defined by

$$\iiint_S \mathbf{f} = \iiint_D \mathbf{f}(\mathbf{r}(s,t,u)) \cdot \Xi\left(\frac{\partial \mathbf{r}}{\partial s}, \frac{\partial \mathbf{r}}{\partial t}, \frac{\partial \mathbf{r}}{\partial u}\right).$$

Finally, we know how to integrate scalar functions over four-dimensional subsets of $\mathbb{R}^4$—such integrals are just ordinary quadruple integrals, and we have dealt with multiple integrals of scalar functions in full generality.

For one-, two-, and three-dimensional integrals, there is a theorem that says the value of a given integral does not depend on the parametrization used (except possibly up to a minus sign). We proved this in class for line integrals in full generality. The idea of the proof for two- or three-dimensional integrals is similar, but depends on having a change-of-variables theorem for double or triple integrals, respectively. (Of course, there is no such theorem for four-dimensional integrals in $\mathbb{R}^4$, since there is no parametrization involved in any multiple scalar integral.)

Epilogue

The differentiation operators we have set up for $\mathbb{R}^4$ are related as shown in the table below. In the course of this discussion, you have seen that integration in $\mathbb{R}^4$ is much more complicated than in $\mathbb{R}^3$, but that there is a pattern. In

$$\begin{array}{c}\text{scalar fields}\\ \big\downarrow{\scriptstyle \text{grad}}\\ \text{vector fields}\\ \big\downarrow{\scriptstyle \text{twist}}\\ \text{skew-symmetric matrix fields}\\ \big\downarrow{\scriptstyle \text{spin}}\\ \text{vector fields}\\ \big\downarrow{\scriptstyle \text{div}}\\ \text{scalar fields}\end{array}$$

Table 1: Differentiation operators in $\mathbb{R}^4$

$\mathbb{R}^5$, there are five differentiation operators, one of which takes a matrix field and gives back another matrix field, but in $\mathbb{R}^6$, we run into trouble. Vector and matrix fields don't suffice for describing integration over three-dimensional subsets of $\mathbb{R}^6$.

The solution is to introduce the idea of *tensors*, an algebraic construction of which vectors and matrices can be thought of as special cases (sometimes called "degree 1" and "degree 2" tensors, respectively). The language of tensors is the starting point of multilinear algebra, just as vectors are the starting point of linear algebra. In this new, general setting, the only kind of function one ever integrates are tensor fields. Even scalar functions can be thought of as "degree zero" tensor fields. (To be more precise, we only integrate "alternating" tensor fields—this corresponds to the fact that we only integrate skew-symmetric, and not arbitrary, matrix fields on two-dimensional surfaces in $\mathbb{R}^4$.)

We also need tensorial replacements for differentiation operators like curl, grad, or twist. Here we get to see how powerful and useful an idea the concept of a tensor actually is: rather than many complicated, tedious, different formulas that must be worked out individually in every dimension, there is just one, unified formula. The result of applying this formula to an alternating tensor field is also an alternating tensor field, but of one degree higher (*cf.* the gradient of a scalar field is a vector field, and the twist of a vector field is a matrix field). This all-encompassing operation is called the "differential" of a tensor field, and it is denoted by the mysterious symbol which we have shunned all year, the lowercase letter d.

With tensors and the differential in our pockets, we can prove a theorem that encompasses the Second Fundamental Theorem and all its generalizations, and extends to arbitrarily many dimensions. It says that if M is a k-dimensional subset of $\mathbb{R}^n$, with a $(k-1)$-dimensional boundary denoted by $\operatorname{Bd} M$, and if ω is an alternating tensor field on M of degree $k-1$, then

$$\int_M d\omega = \int_{\operatorname{Bd} M} \omega.$$

With this glimpse of one of the most important and beautiful theorems of higher-dimensional calculus, our studies draw to a close. I wish each of you the best for your sophomore years and all your future endeavours in life. Thanks for a great year.

Problems

1. State the four integration theorems and three repeated differentiation theorems that hold in $\mathbb{R}^4$.
2. Choose one of the repeated differentiation theorems and prove it.
3. Let $S^3(r) = \{\mathbf{x} \in \mathbb{R}^4 \mid \|\mathbf{x}\| = r\}$, where r is some fixed positive constant. This is called the "3-sphere of radius r."
   (a) Find a unit normal vector field to $S^3(r)$.
   (b) Integrate this vector field over $S^3(r)$. This will give you the 3-dimensional volume (surface area?) of the 3-sphere.
   (c) (Optional) On Problem Set 10 you computed the (4-dimensional) volume of $B^4(r)$. How are these two formulas related? Compare also the formulas for lower-dimensional balls and spheres. Formulate a conjecture about the relationship between the volumes of $B^n(r)$ and $S^{n-1}(r)$ in general.
4. Consider the 2-dimensional surface T in $\mathbb{R}^4$ given by the equations

   $$x^2 + y^2 = 1$$
   $$z^2 + w^2 = 1,$$

   and the matrix field given by

   $$\mathbf{F}(x,y,z,w) = \begin{pmatrix} 0 & 0 & 0 & y/z \\ 0 & 0 & -x/w & 0 \\ 0 & x/w & 0 & 0 \\ -y/z & 0 & 0 & 0 \end{pmatrix}.$$

   Compute $\iint_T \mathbf{F}$.
5. Choose one of the integration theorems in $\mathbb{R}^4$ other than the gradient theorem, and prove it. You may make reasonable simplifying assumptions about the regions of integration and the functions involved, much as we did in our partial proofs of Green's, Stokes' and Gauss' Theorems in class, but be sure to state explicitly what assumptions you make.