18.024–ESG Problem Set 2

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Tuesday

1. (a) Let two lines be given by the following equations:

\[
\begin{align*}
  x &= p + ta \\
  y &= q + sb
\end{align*}
\]

Prove that they intersect if and only if \( p - q \) is in the linear span of \( a \) and \( b \).

(b) Suppose we have a \( k \)-plane and an \( l \)-plane given by the following equations:

\[
\begin{align*}
  x &= p + t_1 a_1 + \cdots + t_k a_k \\
  y &= q + s_1 b_1 + \cdots + s_l b_l
\end{align*}
\]

Prove that they intersect if and only if \( p - q \) is in the linear span of \( \{a_1, \ldots, a_k, b_1, \ldots, b_l\} \).

(c) Determine whether the following 4-plane and 5-plane in \( \mathbb{R}^6 \) intersect.

\[
\begin{align*}
  x &= (7, 4, 6, 8, 5, 4) + t_1 (8, 3, 7, 0, 1, 8) + t_2 (3, 4, 5, 7, 6, 8) \\
  &\quad\quad\quad\quad\quad\quad\quad\quad+ t_3 (2, 0, 1, 8, 4, 3) + t_4 (0, 9, 1, 2, 8, 3) \\
  y &= (3, 4, 5, 7, 6, 8) + s_1 (2, 0, 8, 2, 0, 1) + s_2 (8, 3, 4, 7, 6, 5) \\
  &\quad\quad\quad\quad\quad\quad\quad\quad+ s_3 (0, 1, 9, 2, 0, 3) + s_4 (8, 9, 6, 5, 1, 0) + s_5 (7, 4, 6, 8, 5, 4)
\end{align*}
\]

Thursday

2. Exercise 1 in Section 16.16 of Apostol, Volume I. (Just compute \( B + C, AB \), and \( BA \).)

3. Recall that two matrices \( A \) and \( B \) are said to commute if \( AB = BA \).

   (a) Find all \( 2 \times 2 \) matrices that commute with \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).
(b) Find all $2 \times 2$ matrices that commute with all four of the following matrices:

$$
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(c) (Optional) Prove that a given $2 \times 2$ matrix commutes with all $2 \times 2$ matrices if and only if it commutes with the four matrices listed in Problem 3b. In other words, show that the set of matrices you found in Problem 3b is precisely the set of matrices that commute with all $2 \times 2$ matrices. (In grown-up words, this set of matrices is called the “center of the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$.”)

\textit{Friday}

4. Exercises 1, 2, and 3 in Section 16.16 of Apostol, Volume I.