18.024–ESG Problem Set 2

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Tuesday

1. (a) Let two lines be given by the following equations:

$$\mathbf{x} = \mathbf{p} + t\mathbf{a}$$
$$\mathbf{y} = \mathbf{q} + s\mathbf{b}$$

Prove that they intersect if and only if $\mathbf{p} - \mathbf{q}$ is in the linear span of $\mathbf{a}$ and $\mathbf{b}$.

(b) Suppose we have a k-plane and an l-plane given by the following equations:

$$\mathbf{x} = \mathbf{p} + t_1\mathbf{a}_1 + \cdots + t_k\mathbf{a}_k$$
$$\mathbf{y} = \mathbf{q} + s_1\mathbf{b}_1 + \cdots + s_l\mathbf{b}_l$$

Prove that they intersect if and only if $\mathbf{p} - \mathbf{q}$ is in the linear span of $\{\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_l\}$.

(c) Determine whether the following 4-plane and 5-plane in $\mathbb{R}^6$ intersect.

$$\begin{aligned} \mathbf{x} = (7,4,6,8,5,4) &+ t_1(8,3,7,0,1,8) + t_2(3,4,5,7,6,8) \\ &\quad + t_3(2,0,1,8,4,3) + t_4(0,9,1,2,8,3) \\ \mathbf{y} = (3,4,5,7,6,8) &+ s_1(2,0,8,2,0,1) + s_2(8,3,4,7,6,5) \\ &+ s_3(0,1,9,2,0,3) + s_4(8,9,6,5,1,0) + s_5(7,4,6,8,5,4) \end{aligned}$$

Thursday

2. Exercise 1 in Section 16.16 of Apostol, Volume I. (Just compute $B + C$, AB, and BA.)

3. Recall that two matrices A and B are said to *commute* if $AB = BA$.

   (a) Find all 2×2 matrices that commute with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

(b) Find all 2×2 matrices that commute with all four of the following matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) (Optional) Prove that a given 2×2 matrix commutes with *all* 2×2 matrices if and only if it commutes with the four matrices listed in Problem 3b. In other words, show that the set of matrices you found in Problem 3b is precisely the set of matrices that commute with all 2×2 matrices. (In grown-up words, this set of matrices is called the "center of the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$.")

Friday

4. Exercises 1, 2, and 3 in Section 16.16 of Apostol, Volume I.