18.024–ESG Problem Set 8

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Tuesday

1. Exercises 1 and 2 in Section 10.5 of Apostol, Volume II.

2. Exercise 8 in Section 10.9 of Apostol, Volume II.

Thursday and Friday

3. (a) Show that the vector field $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{f}(x, y) = (x + y)\mathbf{i} + (x - y)\mathbf{j}$$

is conservative, *i.e.*, is the gradient of some scalar field.

(b) Suppose $\mathbf{r} : [a, b] \to \mathbb{R}^2$ is a differentiable curve, say

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}.$$

Compute $\int_a^b \mathbf{f} \cdot d\mathbf{r}$ in terms of $f(a)$, $f(b)$, $g(a)$, and $g(b)$.

4. Determine whether the following vector fields $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ are gradients of scalar fields or not. For each, if it is a gradient, find a scalar field $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla\phi = \mathbf{f}$; if it is not a gradient, explain why it is not.

   (a) $\mathbf{f}(x, y) = y\mathbf{i} - x\mathbf{j}$.

   (b) $\mathbf{f}(x, y) = (3x^2y, x^3)$.

5. Let S denote the set $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, and let $\mathbf{f} : S \to \mathbb{R}^2$ be the vector field

$$\mathbf{f}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

We saw (or will see) in class that $\mathbf{f}$ cannot be the gradient of any scalar field defined on S, because its integral along a certain closed curve is nonzero. (See Example 2 in Section 10.16 of Apostol, Volume II.)

   (a) Exercise 17 in Section 10.18 of Apostol. This exercise shows that $\mathbf{f}$ shares one key property with gradients, *viz.* the equality of "mixed partials."

(b) Exercise 18 in Section 10.18 of Apostol. This exercise has a couple of errors: the set $T \subset \mathbb{R}^2$ should be defined by

$$T = \mathbb{R}^2 \smallsetminus \{(x, y) \mid x = 0 \text{ and } y \leq 0\},$$

i.e., $\mathbb{R}^2$ with the origin and the negative y-axis deleted. Furthermore, θ should be such that $-\pi/2 < \theta < 3\pi/2$. (Alternatively, you could keep Apostol's T and range of values for θ, but then the formula for θ would have to be changed.)

You may be asking yourself, "What's going on here? What is the purpose of looking at this obscure function and seeing that it is a gradient if you cut out part of $\mathbb{R}^2$?" Actually, this problem scratches the surface of a deep and fascinating subject in mathematics called algebraic topology. The point is not to study the function itself, or what it's a gradient of, *etc.* Rather, the point is to look at the domains $\mathbb{R}^2$, S, and T. For a vector field defined on all of $\mathbb{R}^2$, it turns out that equality of mixed partials (which you proved for $\mathbf{f}$ in Exercise 17) is sufficient to guarantee that it is a gradient of some scalar field. The same fact is true on T, but evidently not on S, since we have a counterexample. What's different about S?

Loosely speaking, S has a "hole," whereas neither $\mathbb{R}^2$ nor T does. If you were a topologist, you would say: "I don't care what the vector field $\mathbf{f}$ is; I just care that there exists a vector field with equal mixed partials, yet which is not a gradient. By establishing the existence of such a vector field, I have 'detected' the hole in S!" As humans in real life, we look at S and say, "Well, duh, there is a hole there." But as mathematicians, we find it a good deal more complicated. What do we even mean by the word "hole"? Does $\mathbb{R}^3 \smallsetminus \{\mathbf{0}\}$ have a hole, and if so, is it the same kind of hole that S has? (It turns out that any vector field on $\mathbb{R}^3 \smallsetminus \{\mathbf{0}\}$ with equal mixed partials *is* a gradient, so if there is a hole there, it's fundamentally different from the one in S.) How about $\mathbb{R}^n \smallsetminus \{\mathbf{0}\}$? If you form analogues of T in these higher-dimensional settings, what happens with gradients and vector fields? Do the holes go away, or do they just change "type"?

Some interesting "side theorems" that come out of investigating these questions are Brouwer's Fixed-Point Theorem (which you proved last semester in the one-dimensional case), the Hair-Combing Theorem, and the Ham Sandwich Theorem.