Math 20400: Analysis in $\mathbb{R}^n$ II — Spring 2004

Section 55 — P. Achar

Final Exam Solutions

June 10, 2004

Total points: 200 — Time limit: 2 hours

There are three blank pages for scratch work at the end of the exam.

1. True or False: No justification is required. (2 points each)

   (a) The only compact connected subsets of $\mathbb{R}$ are closed intervals.

   Solution: TRUE

   (b) For subsets of $\mathbb{R}$, being connected is equivalent to being pathwise connected, while in higher dimensions, there are connected sets that are not pathwise connected.

   Solution: TRUE

   (c) Suppose $f : A \to \mathbb{R}$ is a continuous function, where $A \subset \mathbb{R}^n$ is pathwise connected. Then f takes minimum and maximum values.

   Solution: FALSE: this would be true if A were compact.

   (d) Suppose $f : A \to \mathbb{R}$ is continuous, and A is connected. Then $f(A)$ is connected but not necessarily pathwise connected.

   Solution: FALSE: for subsets of $\mathbb{R}$, *connected* and *pathwise connected* are equivalent.

   (e) If $f : A \to \mathbb{R}$ (where $A \subset \mathbb{R}$) is differentiable and $f'(x) = 0$ for all $x \in A$, then f is a constant function.

   Solution: FALSE: this is true if A is an interval, but not if A is not connected.

   (f) If the partial derivatives of $f : A \to \mathbb{R}$ (where $A \subset \mathbb{R}^n$) exist, then f is continuous.

   Solution: FALSE

   (g) The directional derivative of f at $\mathbf{x}_0$ in the direction $\mathbf{p}$ is defined by the equation $\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}_0) = \langle \mathbf{D}f(\mathbf{x}_0), \mathbf{p} \rangle$.

   Solution: FALSE: this equation holds for continuously differentiable functions, in which case it is a theorem, not the definition.

   (h) If $f : A \to \mathbb{R}$ has continuous second-order partial derivatives and a local maximum at $\mathbf{x}_0$, then $\mathbf{D}^2 f(\mathbf{x}_0)$ is negative definite.

   Solution: FALSE

   (i) To say that $f : A \to \mathbb{R}$ "has a tangent hyperplane" at $\mathbf{x}_0$ is equivalent to saying that it has an affine first-order approximation at $\mathbf{x}_0$.

   Solution: TRUE

   (j) If $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there must be an $m \times n$ matrix $\mathbf{A}$ such that $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

   Solution: TRUE

2. Short answer: No justification is required. (4 points each)

   (a) Let U denote the interval $[-1, 1]$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions $f(x) = x^2$ and $g(x) = \sin x$. Find $f(U)$ and $g^{-1}(U)$.

   Solution: $f(U) = [0, 1]$. $g^{-1}(U) = \mathbb{R}$.

(b) Give an example of a function f and a set U such that $f(f^{-1}(U)) \neq U$. Also say exactly what $f^{-1}(U)$ and $f(f^{-1}(U))$ are.

Solution: One possibility is $U = [-1, 4]$ and $f(x) = x^2$. Then $f^{-1}(U) = [-2, 2]$ and $f(f^{-1}(U)) = [0, 4] \neq [-1, 4]$.

(c) Give an example of (i) a connected set that is not compact, and (ii) a compact set that is not connected.

Solution: (i) the open interval $(0, 1)$. (ii) a union of two closed intervals: $[0, 1] \cup [2, 3]$.

(d) Let $f : A \to \mathbb{R}$ (where $A \subset \mathbb{R}^2$) be a function whose second-order partial derivatives exist. Under what conditions can we be sure that $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$?

Solution: This is true if the second-order partials of f are continuous.

(e) State the definition of *limit point*.

Solution: A point $\mathbf{x}_0$ is a limit point of a set A if there is a sequence of points $\{\mathbf{x}_k\}$ in A that converges to $\mathbf{x}_0$, but where $\mathbf{x}_k \neq \mathbf{x}_0$ for all k.

3. Calculations: You should show all your work, but you need not justify every step. (10 points each)

(a) Does the following limit exist? If so, what is the limit?

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^2 + y^4}$$

Solution: The limit is 0.

(b) Find the derivative matrix of the map $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathbf{F}(x, y) = (e^{xy} + 2x, y^2 + \sin(x - y))$.

Solution:

(c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = x^2 + y^2 + z$. Find an equation for the tangent hyperplane to the graph of f at the point $(2, 1, 3, 8)$.

Solution:

(d) Find the local minima and maxima of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (x^2+y^2)e^{x^2+y^2}$.

Proofs: Answer **six** of the following **eight** questions. Unless otherwise specified, you may cite any results from class in your proofs, but you may not cite results that were proved in the homework. (20 points each)

4. Let A be a set that is not compact. Prove that there exists a continuous function $f : A \to \mathbb{R}$ that does not take a maximum value.

Solution: If A is not compact, then it must be either not bounded or not compact. If A is not bounded, then by the definition of boundedness, the function $f : A \to \mathbb{R}$ given by $f(\mathbf{u}) = \|\mathbf{u}\|$ does not attain a maximum value.

If A is not closed, there must be a convergent sequence $\{\mathbf{v}_k\}$ in A whose limit $\mathbf{v}$ does not lie in A. Let us define a function $f : A \to \mathbb{R}$ by the formula $f(\mathbf{u}) = -\|\mathbf{u} - \mathbf{v}\|$. Clearly, f takes only negative values, since $\mathbf{v} \notin A$. However, no negative number is an upper bound for the image of f, since for any negative number $-\epsilon$, there are points of the sequence $\{\mathbf{v}_k\}$ within ϵ of $\mathbf{v}$, so f takes values larger than $-\epsilon$ on those points. In other words, we see that 0 is the least upper bound of the image of f, but since 0 is not in the image, f does not achieve a maximum value. $\square$

(Another possibility for the case where A is not closed is the function $f(\mathbf{u}) = 1/\|\mathbf{u} - \mathbf{v}\|$.)

5. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function. Let V be a closed subset of $\mathbb{R}^m$. Prove that $f^{-1}(V)$ is a closed subset of $\mathbb{R}^n$.

Solution: Let $\{\mathbf{u}_k\}$ be a convergent sequence in $f^{-1}(V)$. We must prove that its limit is also in $f^{-1}(V)$. Suppose that this were not the case. Call its limit $\mathbf{u}$. If $\mathbf{u} \notin f^{-1}(V)$, then $f(\mathbf{u}) \notin V$. By continuity, $\{f(\mathbf{u}_k)\}$ converges to $f(\mathbf{u})$. But this is a sequence in V, which is closed, so $f(\mathbf{u})$ must lie in V after all—a contradiction. $\square$

Alternate solution: Let $W = \mathbb{R}^m \smallsetminus V$: this is an open set. The domain of f (namely, all of $\mathbb{R}^n$) is an open set, so we know that the inverse image $f^{-1}(W)$ is open. We will now show that $f^{-1}(V)$ is the complement of $f^{-1}(W)$; this will establish that $f^{-1}(V)$ is closed. Specifically, we must show that for any point $\mathbf{x} \in \mathbb{R}^n$, we have $\mathbf{x} \in f^{-1}(V)$ if and only if $\mathbf{x} \notin f^{-1}(W)$. First, if $\mathbf{x} \notin f^{-1}(W)$, then $f(\mathbf{x}) \notin W$, which implies $f(\mathbf{x}) \in V$, so $\mathbf{x} \in f^{-1}(V)$. On the other hand, if we start with the assumption that $\mathbf{x} \in f^{-1}(V)$, then we know that $f(\mathbf{x}) \in V$ and so $f(\mathbf{x}) \notin W$, which means that $\mathbf{x} \notin f^{-1}(W)$. Therefore, $\mathbf{x} \in f^{-1}(V)$ if and only if $\mathbf{x} \notin f^{-1}(W)$, so in fact $f^{-1}(V) = \mathbb{R}^n \smallsetminus f^{-1}(W)$, and V is indeed closed. $\square$

Solution:

6. Let $A, B \subset \mathbb{R}^n$ be two pathwise connected sets, and suppose that $A \cap B$ is nonempty. Prove that $A \cup B$ is pathwise connected.

   Solution: Let $\mathbf{u}$ and $\mathbf{v}$ be two points in $A \cup B$. If they are both in A, then we already know that there is a path in $A \cup B$ joining them, since A is pathwise connected. The same holds if they are both in B. The only difficulty is when one is in A and the other in B. Suppose that $\mathbf{u} \in A$ and $\mathbf{v} \in B$. Choose a point $\mathbf{x} \in A \cap B$ (this is possible since $A \cap B$ is nonempty). By the pathwise-connectedness of A we can find a parametrized path $\gamma_1 : [0,1] \to A \cup B$ joining $\mathbf{u}$ to $\mathbf{x}$, and another parametrized path $\gamma_2 : [0,1] \to A \cup B$ joining $\mathbf{x}$ to $\mathbf{v}$. Define a new parametrized path $\gamma : [0,2] \to A \cup B$ by

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0,1] \\ \gamma_2(t-1) & \text{if } t \in [1,2] \end{cases}$$

   Note that this is continuous, since the two parts of the definition agree when $t = 1$. So this is the desired parametrized path joining $\mathbf{u}$ to $\mathbf{v}$. $\square$

7. Let $A, B \subset \mathbb{R}^n$ be two connected sets, and suppose that $A \cap B$ is nonempty. Prove that $A \cup B$ is connected.

   Solution: Suppose that $A \cup B$ is not connected, and let U and V be open sets that separate it. That is, $A \cup B \subset U \cup V$, $U \cap V = \varnothing$, and $(A \cup B) \cap U$ and $(A \cup B) \cap V$ are both nonempty. We will obtain a contradiction by showing that if A is connected, then U and V must separate B. We know that $A \subset U \cup V$. But then, it must be the case that either $A \cap U$ or $A \cap V$ is empty: otherwise, U and V would separate A.

   Suppose $A \cap U$ is empty, so A is contained entirely in V. Since $A \cap B$ is nonempty, B contains at least one point of A, so $B \cap V$ is nonempty. In addition, $B \cap U$ must also be nonempty, since we know that $(A \cup B) \cap U$ is nonempty but A contains no points of U. But if both $B \cap V$ and $B \cap U$ are nonempty, then in fact U and V separate B, contradicting its connectedness.

   Similar reasoning leads to a contradiction if $A \cap V$ is empty instead. $\square$

8. Prove that the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined below is continuously differentiable.

$$g(x,y) = \begin{cases} x^2y^4/(x^2+y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   Solution: We have

$$\frac{\partial g}{\partial x}(x,y) = \frac{2xy^6}{(x^2+y^2)^2}, \qquad \frac{\partial g}{\partial y}(x,y) = \frac{2x^2y^3(2x^2+y^2)}{(x^2+y^2)^2}$$

if $(x, y) \neq (0, 0)$. At $(0, 0)$, we have

$$\frac{\partial g}{\partial x}(0,0) = \lim_{h \to 0} \frac{\frac{(0+h)^2 0^4}{(0+h)^2+0^2} - 0}{h} = 0, \qquad \frac{\partial g}{\partial y}(0,0) = \lim_{h \to 0} \frac{\frac{0^2(0+h)^4}{0^2+(0+h)^2} - 0}{h} = 0.$$

Thus, g has first-order partial derivatives. To show that they are continuous, first observe that $(x^2 + y^2)^2 \geq (y^2)^2 = y^4$. It follows that

$$\left| \frac{2xy^6}{(x^2+y^2)^2} \right| \leq \left| \frac{2xy^6}{y^4} \right| = |2xy^2|.$$

If $\{(x_k, y_k)\}$ converges to $(0, 0)$, then $\lim_{k\to\infty} 2x_k y_k^2 = 0$, so it follows from the above that

$$\lim_{k\to\infty} \frac{2xy^6}{(x^2+y^2)^2} = 0, \qquad \text{so} \qquad \lim_{(x,y)\to(0,0)} \frac{\partial g}{\partial x}(x,y) = \frac{\partial g}{\partial x}(0,0).$$

Thus, $\partial g/\partial x$ is continuous. To show that $\partial g/\partial y$ is continuous, we split it up into two terms:

$$\frac{\partial g}{\partial y}(x,y) = \frac{4x^4y^3}{(x^2+y^2)^2} + \frac{2x^2y^5}{(x^2+y^2)^2}.$$

We then treat the two terms separately: just as above, we have

$$\left| \frac{4x^4y^3}{(x^2+y^2)^2} \right| \leq \left| \frac{4x^4y^3}{x^4} \right| = |4y^3| \qquad \text{and} \qquad \left| \frac{2x^2y^5}{(x^2+y^2)^2} \right| \leq \left| \frac{2x^2y^5}{y^4} \right| = |2x^2y|.$$

From this it follows that

$$\lim_{(x,y)\to(0,0)} \frac{4x^4y^3}{(x^2+y^2)^2} = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(0,0)} \frac{2x^2y^5}{(x^2+y^2)^2} = 0,$$

so $\lim_{(x,y)\to(0,0)} \partial g/\partial y = 0 = \frac{\partial g}{\partial y}(0,0)$. $\frac{\partial g}{\partial y}$ is also continuous, so g is continuously differentiable. $\square$

9. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that $\{x_n\}$ is a strictly increasing, bounded sequence with $f(x_n) \leq f(x_{n+1})$ for all n. Prove that there is a number x_0 at which $f'(x_0) \geq 0$.

   Solution: Since $\{x_n\}$ is a strictly increasing bounded sequence, the Monotone Convergence Theorem says it converges, say to x_0. Since f is differentiable, it is also continuous, so $\{f(x_n)\}$ converges to $f(x_0)$. Moreover, the fact that $f(x_n) \leq f(x_{n+1})$ means that $\{f(x_n)\}$ is an increasing sequence, so in fact $f(x_n) \leq f(x_0)$ for all n. We have

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}.$$

   (Since we know the limit in the middle expression above exists, we can plug in any sequence converging to x_0 for x and obtain a sequence converging to $f'(x_0)$—this is the definition of limit.) Note that both the numerator and denominator of the rightmost expression are less than or equal to 0, so the whole quotient is greater than or equal to 0. Therefore, the limit also satisfies this inequality: $f'(x_0) \geq 0$. $\square$

10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and let $\mathbf{x}_0$ and $\mathbf{h}$ be two fixed points of $\mathbb{R}^n$. Define a function $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(t) = f(\mathbf{x}_0 + t\mathbf{h})$. Prove that $\phi'(t) = \langle \mathbf{D}f(\mathbf{x}_0 + t\mathbf{h}), \mathbf{h} \rangle$.

11. Prove that $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ is pathwise connected.

    Solution: Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be two points in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, and consider the parametrized path $\gamma : [0, 1] \to \mathbb{R}^2$ given by

$$\gamma(t) = (1 - t)\mathbf{u} + t\mathbf{v}.$$

The image of γ is a path from $\mathbf{u}$ to $\mathbf{v}$. We actually want a path in $\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$, so we need to check whether $\mathbf{0}$ is in the image of γ. If $\gamma(t_0) = \mathbf{0}$, it follows that $(1 - t_0)\mathbf{u} = -t_0\mathbf{v}$, or $\mathbf{u} = -\frac{t_0}{1-t_0}\mathbf{v}$, so $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

Thus, if $\mathbf{u}$ and $\mathbf{v}$ are *not* scalar multiples of one another, then γ is a parametrized path in $\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$ that joins them.

What if $\mathbf{u}$ and $\mathbf{v}$ *are* scalar multiples of one another? In this case, we will define a different path passing through a third point. Let $\mathbf{w} = (-u_2, u_1)$. This point is not a multiple of $\mathbf{u}$, and therefore not of $\mathbf{v}$ either. Let $\mu : [0, 2] \to \mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$ be the path

$$\mu(t) = \begin{cases} (1-t)\mathbf{u} + t\mathbf{w} & \text{if } t \in [0,1], \\ (2-t)\mathbf{w} + (t-1)\mathbf{v} & \text{if } t \in [1,2]. \end{cases}$$

Now, μ is a parametrized path joining $\mathbf{u}$ to $\mathbf{v}$. It does not pass through the point $\mathbf{0}$ while $t \in [0, 1]$, since $\mathbf{u}$ and $\mathbf{w}$ are not scalar multiples of one another. Similarly, it does not pass through $\mathbf{0}$ when $t \in [1, 2]$ either. So the image of μ is contained in $\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$.

Since we have found paths in $\mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$ joining any two points in the set, it is pathwise connected. $\square$

12. Let A be a 2×2 matrix that is neither positive definite nor negative definite, and let $Q : \mathbb{R}^2 \to \mathbb{R}$ be the associated quadratic function. Prove that there exists a nonzero vector $\mathbf{x}_0 \in \mathbb{R}^2$ such that $Q(\mathbf{x}_0) = 0$.

Scratch work

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