Final Exam Solutions

June 7, 2004

Total points: 200 — Time limit: 2 hours

There are three blank pages for scratch work at the end of the exam.

1. True or False: No justification is required. (2 points each)

   (a) If $\mathbf{F} : A \to \mathbb{R}^m$ (where $A \subset \mathbb{R}^n$) is stable, then it is one-to-one.

   Solution: TRUE

   (b) Suppose $\mathbf{F} : A \to \mathbb{R}^n$ (where $A \subset \mathbb{R}^n$) is continuously differentiable, and that $\det \mathbf{DF}(\mathbf{x}_0) \neq 0$. Then there is a neighborhood of $\mathbf{x}_0$ in which $\mathbf{F}$ is stable.

   Solution: TRUE: this is one of the steps in the proof of the Inverse Function Theorem.

   (c) The upper and lower Darboux sums are defined for all functions $f : [a, b] \to \mathbb{R}$, integrable or not.

   Solution: FALSE: they are defined for all bounded functions.

   (d) If the Taylor series of f converges at x, then it must necessarily converge to $f(x)$.

   Solution: FALSE

   (e) If $D \subset \mathbb{R}^n$ is bounded and $f : D \to \mathbb{R}$ is continuous, then f is integrable over D.

   Solution: FALSE: this is true if D is a Jordan domain.

2. Short answer: No justification is required. (3 points each)

   (a) Give an example of a set whose volume is not defined.

   Solution: Any unbounded set, such as all of $\mathbb{R}$, has undefined volume. Other examples are bounded sets that are not Jordan domains, such as the set $\{x \in [0, 1] \mid x \text{ is rational}\}$.

   (b) State the definition of *Jordan domain*.

   Solution: A set $D \subset \mathbb{R}^n$ is a Jordan domain if its boundary ∂D has Jordan content zero.

   (c) Give an example of an integrable function that is not continuous.

   Solution: One possibility is $f : [-1, 1] \to \mathbb{R}$ given by

   $$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

   (d) Give an example of a function $f : [a, b] \to \mathbb{R}$ such that the function $F(x) = \int_a^x f$ is not differentiable.

   Solution: The function f given above for part (d) is such a function. For this f, one can verify that $F(x) = |x| - 1$, which is not differentiable at 0.

   (e) State the Integrability Criterion for functions defined on a generalized rectangle.

   Solution: A function $f : \mathbf{I} \to \mathbb{R}$ is integrable if and only if for any $\epsilon > 0$, there exists a partition $\mathbf{P}$ of $\mathbf{I}$ such that $U(f, \mathbf{P}) - L(f, \mathbf{P}) < \epsilon$.

Calculations: You should show all your work, but you need not justify every step. (25 points each)

3. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the function given by $\mathbf{F}(x, y) = (e^{xy} + 2x, y^2 + \sin(x - y))$.

(a) Find the derivative matrix of $\mathbf{F}$.

Solution:

$$\mathbf{DF}(x,y) = \begin{bmatrix} ye^{xy}+2 & xe^{xy} \\ \cos(x-y) & 2y-\cos(x-y) \end{bmatrix}$$

(b) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and let $h = g \circ \mathbf{F}$. Find $\partial h/\partial x$ in terms of the partial derivatives of g.

Solution: By the chain rule, $\mathbf{D}h(\mathbf{x}) = \mathbf{D}g(\mathbf{F}(\mathbf{x}))\mathbf{DF}(\mathbf{x})$. That is,

$$\begin{bmatrix} \frac{\partial h}{\partial x}(x,y) & \frac{\partial h}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x}(\mathbf{F}(x,y)) & \frac{\partial h}{\partial y}(\mathbf{F}((x,y)) \end{bmatrix} \begin{bmatrix} ye^{xy}+2 & xe^{xy} \\ \cos(x-y) & 2y-\cos(x-y) \end{bmatrix}.$$

In particular,

$$\begin{aligned} \frac{\partial h}{\partial x}(x,y) &= \frac{\partial g}{\partial x}(\mathbf{F}(x,y))(ye^{xy}+2) + \frac{\partial g}{\partial y}(\mathbf{F}(x,y))\cos(x-y) \\ &= (ye^{xy}+2)\frac{\partial g}{\partial x}(e^{xy}+2x, y^2+\sin(x-y)) + \cos(x-y)\frac{\partial g}{\partial y}(e^{xy}+2x, y^2+\sin(x-y)). \end{aligned}$$

(c) At which of the following points does the Inverse Function Theorem apply to $\mathbf{F}$? For each point $\mathbf{x}$ where it applies, there is a neighborhood of $\mathbf{x}$ in which $\mathbf{F}$ has an inverse function $\mathbf{G}$. The Inverse Function Theorem tells you how to compute $\mathbf{DG}$ at a particular point $\mathbf{y}$. Say what $\mathbf{y}$ is, and compute $\mathbf{DG}(\mathbf{y})$.

$$(0,0), \qquad (\pi/2,0), \qquad (0,-2)$$

Solution: Plugging in these points to the derivative matrix, we obtain

$$\mathbf{DF}(0,0) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{DF}(\pi/2,0) = \begin{bmatrix} 2 & \pi/2 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{DF}(0,-2) = \begin{bmatrix} 0 & 0 \\ \cos 2 & -4-\cos 2 \end{bmatrix}.$$

The determinants of these matrices are:

$$\det \mathbf{DF}(0,0) = -2, \qquad \det \mathbf{DF}(\pi/2,0) = 0, \qquad \det \mathbf{DF}(0,-2) = 0,$$

so the Inverse Function Theorem applies at $(0,0)$ and not at the other two points. Now, note that $\mathbf{F}(0,0) = (1,0)$. If $\mathbf{G}$ is the inverse function to $\mathbf{F}$ in some neighborhood of $(0,0)$, then $\mathbf{G}(1,0) = (0,0)$. According to the Inverse Function Theorem,

$$\begin{aligned} \mathbf{DG}(1,0) &= [\mathbf{DF}((G(1,0)))]^{-1} = [\mathbf{DF}(0,0)]^{-1} \\ &= \begin{bmatrix} 2 & 0 & 1 & -1 \end{bmatrix}^{-1} \\ &= \frac{1}{-2}\begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix}. \end{aligned}$$

4. Let $D \subset \mathbb{R}^2$ be the following triangular region:

$$D = \{(x,y) \in [0,1] \times [0,1] \mid y \leq x\}.$$

Let Γ be the boundary of D, regarded as a path parametrized in the counterclockwise direction.

(a) Find a parametrization of Γ. (*Hint*: Your answer will probably have three separate parts.)

Solution: The region is a triangle with a vertical edge going from $(1,0)$ to $(1,1)$, then a diagonal edge going from $(1,1)$ to $(0,0)$, and finally a horizontal edge going from $(0,0)$ to $(1,0)$. One parametrization of this path is as follows: define $\gamma : [0,3] \to \mathbb{R}^2$ by

$$\gamma(t) = \begin{cases} (1,t) & \text{if } 0 \leq t \leq 1, \\ (2-t, 2-t) & \text{if } 1 \leq t \leq 2, \\ (t-2, 0) & \text{if } 2 \leq t \leq 3. \end{cases}$$

(b) Using your parametrization, evaluate the following line integral:
$\int_\Gamma x^3\,dy + y^2\,dx$

Solution:

$$\begin{aligned}
\int_\Gamma x^3\,dy + y^2\,dx &= \int_0^3 ((\gamma_1(t))^3\gamma_2'(t) + (\gamma_2(t))^2\gamma_1'(t))\,dt \\
&= \int_0^1 ((\gamma_1(t))^3\gamma_2'(t) + (\gamma_2(t))^2\gamma_1'(t))\,dt + \int_1^2 ((\gamma_1(t))^3\gamma_2'(t) + (\gamma_2(t))^2\gamma_1'(t))\,dt \\
&\quad + \int_2^3 ((\gamma_1(t))^3\gamma_2'(t) + (\gamma_2(t))^2\gamma_1'(t))\,dt \\
&= \int_0^1 (1^3\cdot 1 + t^2\cdot 0)\,dt + \int_1^2 ((2-t)^3\cdot(-1) + (2-t)^2\cdot(-1))\,dt \\
&\quad + \int_2^3 ((t-2)^3\cdot 0 + 0^2\cdot 1)\,dt
\end{aligned}$$

Substituting $u = 2-t$ in the second integral, we obtain

$$\begin{aligned}
&= \int_0^1 1 + \int_1^0 (u^3+u^2)\,du \\
&= 1 + \left(\frac{u^4}{4} + \frac{u^3}{3}\right)\Big|_1^0 = 1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12}.
\end{aligned}$$

(c) Evaluate the same line integral using Green's Theorem.
Solution: Let $M(x,y) = x^3$ and $N(x,y) = y^2$. According to Green's Theorem,

$$\int_\Gamma x^3\,dy + y^2\,dx = \int_D \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) = \int_D (3x^2 - 2y)$$

By Fubini's Theorem, we get:

$$\begin{aligned}
&= \int_0^1 \left(\int_0^x (3x^2-2y)\,dy\right)\,dx \\
&= \int_0^1 (3x^2y - y^2)\big|_0^x\,dx = \int_0^1 (3x^3 - x^2)\,dx \\
&= \left(\frac{3x^4}{4} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}.
\end{aligned}$$

5. Let $r > 0$ be a constant, and let $\mathcal{S}$ be the set

$$\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \mid x,y,z > 0 \text{ and } z^2 = r^2 - x^2 - y^2\}.$$

Find a parametrization of this surface, and then find its surface area. (Significant partial credit will be given if you can express the surface area as an ordinary 2-dimensional integral over some region in $\mathbb{R}^2$, even if you don't actually compute the value of the integral.)

Solution: In view of the equation $z^2 = r^2 - x^2 - y^2$, we see that $r^2 - x^2 - y^2$ must be nonnegative, so $x^2 + y^2 \le r^2$. To parametrize $\mathcal{S}$, we can take the domain D to be the quarter-disk

$$D = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0, \text{ and } x^2 + y^2 \le r^2\},$$

with the actual parametrization given by

$$\mathbf{r} : D \to \mathbb{R}^3, \qquad \mathbf{r}(x, y) = (x, y, \sqrt{r^2 - x^2 - y^2}).$$

(We know to take the positive square root since $z > 0$ for points in $\mathcal{S}$.) Then we have

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial x} &= \left(1, 0, -\frac{x}{\sqrt{r^2 - x^2 - y^2}}\right) \\
\frac{\partial \mathbf{r}}{\partial y} &= \left(0, 1, -\frac{y}{\sqrt{r^2 - x^2 - y^2}}\right) \\
\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} &= \left(\frac{x}{\sqrt{r^2 - x^2 - y^2}}, \frac{x}{\sqrt{r^2 - x^2 - y^2}}, 1\right) \\
\left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| &= \sqrt{\frac{x^2}{r^2 - x^2 - y^2} + \frac{y^2}{r^2 - x^2 - y^2} + 1} = \frac{r}{\sqrt{r^2 - x^2 - y^2}}.
\end{aligned}$$

Therefore, the area of $\mathcal{S}$ is given by

$$\int_D \left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| = \int_0^r \left(\int_0^{\sqrt{r^2 - x^2}} \frac{r\,dy}{\sqrt{r^2 - x^2 - y^2}} \right) dx$$

Let us introduce the trigonometric substitution $y = \sqrt{r^2 - x^2}\sin\theta$, so $dy = \sqrt{r^2 - x^2}\cos\theta\,d\theta$:

$$\begin{aligned}
&= \int_0^r \left(\int_0^{\pi/2} \frac{r\sqrt{r^2 - x^2}\cos\theta\,d\theta}{\sqrt{r^2 - x^2}\sqrt{1 - \sin^2\theta}} \right) dx \\
&= \int_0^r \left(\int_0^{\pi/2} r\,d\theta \right) dx \\
&= \int_0^r \frac{\pi r}{2}\,dx = \frac{\pi r^2}{2}.
\end{aligned}$$

Proofs: Answer **five** of the following **six** questions. Unless otherwise specified, you may cite any results from class in your proofs, but you may not cite results that were proved in the homework. (20 points each)

6. Suppose that $\mathbf{F} : A \to \mathbb{R}^m$ (where $A \subset \mathbb{R}^n$ is closed) is continuously differentiable and stable. Prove that $\mathbf{F}(A)$ is closed.

   Solution: Let $\{\mathbf{u}_k\}$ be a convergent sequence in $\mathbf{F}(A)$, converging to $\mathbf{u}$. We need to show that $\mathbf{u} \in \mathbf{F}(A)$. Since each $\mathbf{u}_k$ is in the image of $\mathbf{F}$, we can choose a point $\mathbf{v}_k \in A$ such that $\mathbf{F}(\mathbf{v}_k) = \mathbf{u}_k$. These points form a new sequence $\{\mathbf{v}_k\}$, which may or may not converge. But in this case, we can use the fact that $\mathbf{F}$ is stable to show that it does.

   Specifically, let c be the stability constant for $\mathbf{F}$: we have

   $$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \geq c\|\mathbf{x} - \mathbf{y}\|.$$

   Since $\{\mathbf{u}_k\}$ converges, it is a Cauchy sequence. Given $\epsilon > 0$, let us apply the definition of "Cauchy sequence" with respect to $c\epsilon$: there is an integer $N > 0$ such that if $n, m \geq N$, then $\|\mathbf{u}_n - \mathbf{u}_m\| < c\epsilon$. That is, $\|\mathbf{F}(\mathbf{v}_n) - \mathbf{F}(\mathbf{v}_m)\| < c\epsilon$. By stability, it follows that

   $$c\|\mathbf{v}_n - \mathbf{v}_m\| < c\epsilon,$$

   or simply $\|\mathbf{v}_n - \mathbf{v}_m\| < \epsilon$ for $n, m \geq N$. We have proved that $\{\mathbf{v}_k\}$ is Cauchy, and hence convergent. Let $\mathbf{v}$ be its limit. By continuity, $\{\mathbf{F}(\mathbf{v}_k)\}$ (that is, $\{\mathbf{u}_k\}$) converges to $\mathbf{F}(\mathbf{v})$. So $\mathbf{u} = \mathbf{F}(\mathbf{v})$ is in $\mathbf{F}(A)$, and the image of $\mathbf{F}$ is closed. $\square$

7. Suppose that the function $f : [0, \infty) \to [0, \infty)$ is continuous and stable, and that $f(0) = 0$. Prove that f is onto.

   Solution: Let c be the stability constant for f. Given $y \in [0, \infty)$, we must prove that y is in the image of f. Let us evaluate f at y/c and 0: stability says that

   $$|f(y/c) - f(0)| \geq c|y/c - 0| = |y|.$$

   Since we know that $f(0) = 0$, this inequality just becomes $|f(y/c)| \geq |y|$. We can drop the absolute value signs on the right since $y \geq 0$. In fact, we can do so on the left, too, since the codomain of f is $[0, \infty)$—this means that $f(x) \geq 0$ for all x. So we have $f(y/c) \geq y$. Again using the fact that $f(0) = 0$, we have

   $$f(0) \leq y \leq f(y/c).$$

   Therefore, by the Intermediate Value Theorem, there is a point x between 0 and y/c such that $f(x) = y$. Thus, f is onto. ☐

8. Using the Lagrange Remainder Theorem, prove that

   $$1 + \frac{x}{3} - \frac{x^2}{9} < (1 + x)^{1/3} < 1 + \frac{x}{3}$$

   if $x > 0$.

   Solution: Define $f : (-1, \infty) \to \mathbb{R}$ by $f(x) = \sqrt[3]{1 + x}$. Then $f'(x) = \frac{1}{3}(1 + x)^{-2/3}$ and $f''(x) = -\frac{2}{9}(1 + x)^{-5/3}$. The first Taylor polynomial for f around $x_0 = 0$ is

   $$p_1(x) = f(0) + f'(0)x = 1 + \frac{x}{3}.$$

   According to the Lagrange Remainder Theorem, for each x, there is some c between 0 and x such that

   $$f(x) - p_1(x) = \frac{f''(c)}{2}x^2 = -\frac{1}{9(1 + c)^{5/3}}x^2.$$

   Now, if $x > 0$, then c must be positive as well (since it's between x and 0), so the entire expression $-x^2/9(1 + c)^{5/3}$ is negative. That is, $f(x) - p_1(x) < 0$. On the other hand, since c is positive, it is clear that $(1 + c)^{5/3} > 1$, so $x^2/9(1 + c)^{5/3} < x^2/9$. Putting this together, we have

   $$-x^2/9 < f(x) - p_1(x) = \sqrt[3]{1 + x} - (1 + x/3) < 0,$$

   so

   $$1 + \frac{x}{3} - \frac{x^2}{9} < \sqrt[3]{1 + x} < 1 + \frac{x}{3}$$

   for $x > 0$, as desired. ☐

9. Let $D \subset \mathbb{R}^n$ be a Jordan domain, and suppose that $f : D \to \mathbb{R}$ is an integrable function with the property that $f(\mathbf{x}) = 0$ if $\mathbf{x}$ has at least one rational coordinate. Prove that $\int_D f = 0$.

   Solution: Let $\mathbf{I}$ be a rectangle containing D, and let $\mathbf{P}$ be a partition of $\mathbf{I}$.

10. Let B_r^n denote the set $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq r\}$. Prove, using Fubini's Theorem, that $\text{vol}(B_r^4) = \frac{1}{2}\pi^2 r^4$. (You may use without proof the fact that $\text{vol}(B_r^3) = \frac{4}{3}\pi r^3$, as well as the fact that B_r^3 and B_r^4 are both Jordan domains.)

11. Let E denote the set

    $$E = \{(x, y) \in \mathbb{R}^2 \mid x, y > 0 \text{ and } x + y = 1\}.$$

    Prove that E has Jordan content zero. (For this problem, you may not use the theorem from the textbook which asserts that the graph of a function has Jordan content zero.)

Scratch work

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