Extra Credit

Due: December 6, 2005

The *dimension* of a geometric object is the number of coordinates it takes to specify a location in the object. For example, the surface of a sphere (such as the surface of the earth) is 2-dimensional, because you can specify a location with two numbers: its latitude and longitude. On the other hand, a solid sphere is 3-dimensional: to identify a point *within* the earth, you must give its latitude, longitude, and depth from the surface. The way in which one speaks of the "size" of an object depends on its dimension: 1-dimensional objects have length, 2-dimensional things have area, and 3-dimensional things have volume.

It is possible to study the geometry of objects with more than three dimensions, even though we obviously can't imagine or draw pictures of such objects. There aren't any ordinary English words to describe things in more than three dimensions, however, so what mathematicians usually do is just use 3-dimensional terminology, but with added numerical prefixes to indicate the dimension. For instance, there is no English word for the "size" of a 4-dimensional object (which would be measured in cm^4 or in^4), so mathematicians just call it "4-volume." You can also use "1-volume" as a synonym for "length," and "2-volume" as a synonym for "area."

Circles and spheres are part of a family of shapes in all different dimensions. The goal of this problem is to find the 4-volume of the 4-dimensional member of this family. In general, an *n-sphere of radius r* is the shape you get by starting at the origin in n-dimensional space and going out a distance of r in "all directions" (what that means depends on what dimension you're working in, of course)[1]. You already know formulas for their sizes in low dimensions:

Object	*Common name*	*Type of "size"*	*Formula for "size"*
1-sphere of radius r	line segment of length $2r$	1-volume, or length	$2r$
2-sphere of radius r	circle of radius r	2-volume, or area	πr^2
3-sphere of radius r	ordinary sphere of radius r	3-volume, or ordinary volume	$\frac{4}{3}\pi r^3$

(Make sure you understand why a 1-sphere of radius r is the same as a line segment of length $2r$.)

Another way of describing spheres of various dimensions is by stacking up spheres of smaller dimensions. (You can make a circle by stacking up line segments of varying lengths, and you can make an ordinary (3-) sphere by stacking up circles.) We can use this description to find volume formulas.

For example, to draw a circle of radius r in the plane, you go from $x = -r$ to $x = r$, and at each x-value, you draw a little vertical line segment whose length is $2\sqrt{r^2 - x^2}$ (think about why this is true—it comes from the equation for a circle). Then, the area of the circle should be

$$\int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx.$$

If you work out this integral, you get πr^2 as the answer. Note that the quantity inside the integral is the 1-volume of a 1-sphere of radius $\sqrt{r^2 - x^2}$.

Next, to make an ordinary 3-sphere of radius r in 3-dimensional space, you go from $x = -r$ to $x = r$, and at each x-value, you draw a vertical circle whose radius is $\sqrt{r^2 - x^2}$. The area of one of these circular cross sections (*i.e.*, the 2-volume of a 2-sphere of radius $\sqrt{r^2 - x^2}$) is $\pi(\sqrt{r^2 - x^2})^2$, so the volume of the whole sphere is

$$\int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2\,dx.$$

We worked out this integral in class, and it comes out to $\frac{4}{3}\pi r^3$.

[1] This isn't quite the standard terminology. Mathematicians usually use the word "sphere" to refer to the boundary of this shape, not including its interior, so the dimension in the above definition is off by one from the standard terminology.

Problems

1. (3 points) A 4-sphere is made by stacking up cross-sections that are 3-spheres. Following the pattern from 2- and 3-spheres, set up an integral for the 4-volume of a 4-sphere of radius r.

2. (2 points) Find the formula for the 4-volume of a 4-sphere by evaluating the integral you have set up. (Doing this from scratch requires techniques that we haven't learned yet, but you can do it using the integral tables in the back of the textbook.)

3. (Extra Extra Credit—2 points) Find the 5-volume of a 5-sphere of radius r. (Since the cross-sections of a 5-sphere are 4-spheres, you need to have the correct formula for the 4-volume of a 4-sphere in order to do this problem.)

4. (Super Ultra Extra Credit—5 points) Find a general formula (in terms of n) for the n-volume of an n-sphere. Actually, it turns out that the volumes of even-dimensional spheres follow one pattern, and those of odd-dimensional spheres follow another. So you should actually find *two* formulas: one that's valid when n is even, and the other when n is odd.