Math 1550 | Fall 2005

Section 31 | P. Achar

Extra Credit Solutions

Due: December 6, 2005

1. (3 points) A 4-sphere is made by stacking up cross-sections that are 3-spheres. Following the pattern from 2- and 3-spheres, set up an integral for the 4-volume of a 4-sphere of radius r.

   Solution: Imagine a 4-sphere of radius r centered at the origin in 4-dimensional space. If you take a "slice" perpendicular to the x-axis, you get a 3-sphere. Moreover, the radius of that slice is $\sqrt{r^2 - x^2}$. The possible values of x are from $-r$ to r, so to compute the 4-volume by cross-sections, we set up the integral:

   $$\int_{-r}^{r} \left(\begin{array}{c} \text{3-volume of the} \\ \text{cross-section} \end{array} \right) dx = \int_{-r}^{r} \left(\begin{array}{c} \text{3-volume of a 3-sphere} \\ \text{of radius } \sqrt{r^2 - x^2} \end{array} \right) dx$$
   $$= \int_{-r}^{r} \frac{4}{3}\pi(\sqrt{r^2 - x^2})^3 \, dx$$

2. (2 points) Find the formula for the 4-volume of a 4-sphere by evaluating the integral you have set up. (Doing this from scratch requires techniques that we haven't learned yet, but you can do it using the integral tables in the back of the textbook.)

   Solution:
   $$\int_{-r}^{r} \frac{4}{3}\pi(\sqrt{r^2 - x^2})^3 \, dx$$
   $$= \frac{4}{3}\pi \int_{-r}^{r} (r^2 - x^2)^{3/2} \, dx$$

   for the next step, use formula #37 from the integral tables at the back of the book

   $$= \frac{4}{3}\pi \left(-\frac{x}{8}(2x^2 - 5r^2)\sqrt{r^2 - x^2} + \frac{3r^4}{8} \sin^{-1} \frac{x}{r} \right) \Big|_{-r}^{r}$$
   $$= \frac{4}{3}\pi \left(\left(-\frac{r}{8}(2r^2 - 5r^2)\sqrt{r^2 - r^2} + \frac{3r^4}{8} \sin^{-1} \frac{r}{r} \right) - \left(\frac{r}{8}(2r^2 - 5r^2)\sqrt{r^2 - r^2} + \frac{3r^4}{8} \sin^{-1} \frac{-r}{r} \right) \right)$$
   $$= \frac{4}{3}\pi \left(\frac{3r^4}{8} \sin^{-1} 1 - \frac{3r^4}{8} \sin^{-1}(-1) \right)$$
   $$= \frac{4}{3}\pi \cdot \frac{3r^4}{8} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{4}{3}\pi \cdot \frac{3r^4}{8}\pi$$
   $$= \frac{1}{2}\pi^2 r^4.$$

3. (Extra Extra Credit—2 points) Find the 5-volume of a 5-sphere of radius r. (Since the cross-sections of a 5-sphere are 4-spheres, you need to have the correct formula for the 4-volume of a 4-sphere in order to do this problem.)

   Solution: The set-up is the same as in lower-dimensions: we find the 5-volume by integrating the 4-volume of 4-dimensional cross-sections. The cross-section at a given value of x is a 4-sphere whose radius is $\sqrt{r^2 - x^2}$.

$$\int_{-r}^{r} \frac{1}{2}\pi^2(\sqrt{r^2-x^2})^4\,dx$$

$$= \frac{1}{2}\pi^2 \int_{-r}^{r} (r^2-x^2)^2\,dx = \frac{1}{2}\pi^2 \int_{-r}^{r} (r^4-2r^2x^2+x^4)\,dx$$

$$= \frac{1}{2}\pi^2 \left(r^4x - \frac{2r^2x^3}{3} + \frac{x^5}{5}\right)\Big|_{-r}^{r}$$

$$= \frac{1}{2}\pi^2 \left(\left(r^5 - \frac{2r^5}{3} + \frac{r^5}{5}\right) - \left(-r^5 + \frac{2r^5}{3} - \frac{r^5}{5}\right)\right) = \frac{1}{2}\pi^2\left(\frac{16}{15}r^5\right)$$

$$= \frac{8}{15}\pi^2 r^5.$$

4. (Super Ultra Extra Credit—5 points) Find a general formula (in terms of n) for the n-volume of an n-sphere. Actually, it turns out that the volumes of even-dimensional spheres follow one pattern, and those of odd-dimensional spheres follow another. So you should actually find *two* formulas: one that's valid when n is even, and the other when n is odd.

Solution: The key to deriving the formulas in the general case is to go up two dimensions at once. This turns out to be more or less a "volume by cylindrical shells" problem, much as the previous parts were "volume by cross-sections." Imagine an n-sphere of radius r centered at the origin in n-dimensional space. In the earlier problems, you used the fact that the cross-sections perpendicular to the x-axis are $(n-1)$-spheres of various radii. Similarly, a sliver of the n-sphere that's perpendicular to the entire xy-plane is an $(n-2)$-sphere.

Going the other way, imagine attaching an $(n-2)$-sphere to each point on the positive x-axis from $x=0$ to $x=r$, in such a way that it is perpendicular to the xy-plane. (Try imagining it in the case $n=3$ if you have difficulty with the general case.) The radius of the $(n-2)$-sphere attached at a given x-value should be $\sqrt{r^2-x^2}$. This gives you an $(n-1)$-dimensional solid. Now, form a "solid of revolution" by rotating this shape around on the xy-plane. The resulting object is the n-sphere.

Let $V_n(r)$ denote the n-volume of the n-sphere of radius r. The n-dimensional version of the cylindrical shells formula says that

$$V_n(r) = \int_0^r 2\pi x \cdot \Big((n-2)\text{-volume of the } (n-2)\text{-sphere attached at } x\Big)\,dx$$

$$= \int_0^r 2\pi x\, V_{n-2}(\sqrt{r^2-x^2})\,dx.$$

Now, we know that $V_{n-2}(r)$ is of the form ar^{n-2}, where a is some constant. Now, we can evaluate this integral by substitution:

$$V_n(r) = \int_0^r 2\pi x \cdot a(\sqrt{r^2-x^2})^{n-2}\,dx \qquad \text{substitute } u = r^2-x^2,\ du = -2x\,dx$$

$$= \int_{x=0}^{x=r} \pi a(\sqrt{u})^{n-2}\cdot -du \qquad \text{change endpoints of the integral:}$$

$$= -\int_{r^2}^{0} \pi a u^{\frac{n-2}{2}}\,du \qquad x=0 \Longrightarrow u=r^2 \text{ and } x=r \Longrightarrow u=0$$

$$= -\frac{\pi a u^{\frac{n-2}{2}+1}}{\frac{n-2}{2}+1}\Bigg|_{r^2}^{0} = -\frac{\pi a u^{n/2}}{n/2}\Bigg|_{r^2}^{0} \qquad \text{since } \frac{n-2}{2}+1 = \frac{n}{2}$$

$$= -\left(0 - \frac{\pi a (r^2)^{n/2}}{n/2}\right) = \frac{\pi a r^n}{n/2}$$

$$= \frac{\pi r^2}{n/2}\cdot a r^{n-2} = \frac{\pi r^2}{n/2} V_{n-2}(r) = \frac{2\pi r^2}{n} V_{n-2}(r).$$

This relationship lets us easily work out many more low-dimensional sphere volumes. We'll use the form $V_n(r) = \frac{2\pi r^2}{n} V_{n-2}(r)$ for odd dimensions, and $V_n(r) = \frac{\pi r^2}{n/2} V_{n-2}(r)$ for even dimensions:

Dimension	Volume	Dimension	Volume
1	$2r$	2	πr^2
3	$\frac{2\pi r^2}{3} \cdot 2r = \frac{4}{3}\pi r^3$	4	$\frac{\pi r^2}{2} \cdot \pi r^2 = \frac{1}{2}\pi^2 r^4$
5	$\frac{2\pi r^2}{5} \cdot \frac{2\pi r^2}{3} \cdot 2r = \frac{8}{15}\pi^2 r^5$	6	$\frac{\pi r^2}{3} \cdot \frac{\pi r^2}{2} \cdot \pi r^2 = \frac{1}{6}\pi^3 r^6$
7	$\frac{2\pi r^2}{7} \cdot \frac{2\pi r^2}{5} \cdot \frac{2\pi r^2}{3} \cdot 2r = \frac{16}{105}\pi^3 r^7$	8	$\frac{\pi r^2}{4} \cdot \frac{\pi r^2}{3} \cdot \frac{\pi r^2}{2} \cdot \pi r^2 = \frac{1}{24}\pi^4 r^8$

Now, we can write down the general formulas. If n is even, then

$$V_n(r) = \frac{1}{2 \cdot 3 \cdot 4 \cdots (n/2)} \pi^{n/2} r^n = \frac{1}{(n/2)!} \pi^{n/2} r^n.$$

If n is odd, the calculation is somewhat harder:

$$\begin{aligned}
V_n(r) &= \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{2}{5} \cdots \frac{2}{n} \pi^{\frac{n-1}{2}} r^n \\
&= 2^{\frac{n+1}{2}} \frac{1}{1 \cdot 3 \cdot 5 \cdots n} \pi^{\frac{n-1}{2}} r^n = 2^{\frac{n+1}{2}} \frac{2 \cdot 4 \cdot 6 \cdots (n-1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots (n-1) \cdot n} \pi^{\frac{n-1}{2}} r^n \\
&= 2^{\frac{n+1}{2}} \frac{(2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdots (2 \cdot \frac{n-1}{2})}{n!} \pi^{\frac{n-1}{2}} r^n = 2^{\frac{n+1}{2}} \frac{2^{\frac{n-1}{2}} \cdot 1 \cdot 2 \cdot 3 \cdots \frac{n-1}{2}}{n!} \pi^{\frac{n-1}{2}} r^n \\
&= \frac{2^{\frac{n+1}{2} + \frac{n-1}{2}} (\frac{n-1}{2})!}{n!} \pi^{\frac{n-1}{2}} r^n.
\end{aligned}$$

Since $\frac{n+1}{2} + \frac{n-1}{2} = n$, the final answer is:

$$V_n(r) = \begin{cases} \frac{1}{(n/2)!} \pi^{n/2} r^n & \text{if } n \text{ is even,} \\ \\ \frac{2^n (\frac{n-1}{2})!}{n!} \pi^{(n-1)/2} r^n & \text{if } n \text{ is odd.} \end{cases}$$

(In reality, the above work is still a guess, not a mathematically rigorous proof. However, we do now have a known formula for going up two dimensions at once. Using that and a technique called "mathematical induction," it's not hard to turn this into a rigorous proof.)