Extra Credit Solution

Due: October 25, 2005

There is more than one way to find the formula. Here is a solution using logarithmic differentiation:

$$\begin{aligned}
y &= f(x)^{g(x)} \\
\ln y &= \ln f(x)^{g(x)} = g(x)\ln f(x) && \text{(from logarithm laws)} \\
\frac{1}{y}\frac{dy}{dx} &= g(x)\frac{d}{dx}\ln f(x) + \ln f(x)\frac{d}{dx}g(x) && \text{chain rule; product rule} \\
&= g(x)\frac{1}{f(x)}f'(x) + \ln f(x)\cdot g'(x) && \text{chain rule} \\
\frac{dy}{dx} &= y\left(\frac{g(x)f'(x)}{f(x)} + \ln f(x)\cdot g'(x)\right) \\
&= f(x)^{g(x)}\left(\frac{g(x)f'(x)}{f(x)} + \ln f(x)\cdot g'(x)\right) \\
&= g(x)\frac{f(x)^{g(x)}}{f(x)}f'(x) + f(x)^{g(x)}\ln f(x)\cdot g'(x) \\
\frac{d}{dx}\left(f(x)^{g(x)}\right) &= g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}\ln f(x)\cdot g'(x).
\end{aligned}$$

Here is a different solution:

$$\begin{aligned}
\frac{d}{dx}\left(f(x)^{g(x)}\right) &= \frac{d}{dx}\left(\left(e^{\ln f(x)}\right)^{g(x)}\right) && \text{since } f(x) = e^{\ln f(x)} \\
&= \frac{d}{dx}\left(e^{\ln f(x)\cdot g(x)}\right) \\
&= e^{\ln f(x)\cdot g(x)}\frac{d}{dx}(\ln f(x)\cdot g(x)) && \text{chain rule} \\
&= \left(e^{\ln f(x)}\right)^{g(x)}\left(\ln f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}\ln f(x)\right) && \text{product rule} \\
&= f(x)^{g(x)}\left(\ln f(x)\cdot g'(x) + g(x)\cdot\frac{1}{f(x)}\frac{d}{dx}f(x)\right) && \text{chain rule} \\
&= f(x)^{g(x)}\left(\ln f(x)\cdot g'(x) + \frac{g(x)f'(x)}{f(x)}\right) \\
&= f(x)^{g(x)}\ln f(x)\cdot g'(x) + g(x)\frac{f(x)^{g(x)}}{f(x)}f'(x) \\
\frac{d}{dx}\left(f(x)^{g(x)}\right) &= g(x)f(x)^{g(x)-1}f'(x) + f(x)^{g(x)}\ln f(x)\cdot g'(x).
\end{aligned}$$

Now that we have the formula, we can make the following somewhat surprising observation about it: it's the sum of two terms, one of which looks like the old Power Rule, and the other of which looks like the old Exponential Rule. This is not just a coincidence, but we don't have the techniques yet to explain why this is the case. (It's a consequence of the "multivariable chain rule" that you'll learn in Calculus III. Using that, you can deduce the above formula in one step, rather than seven or eight.)

Some of you wrote down examples using the formula. That's fine as a method of convincing yourself that you haven't made a mistake, but examples alone cannot be a mathematical proof of anything. The only way to prove that a formula is correct is to deduce it abstractly, as above, without making any specific choices about what $f(x)$ and $g(x)$ are.