Math 1550 — Fall 2005

Section 31 — P. Achar

Exam 1 Solutions

September 15, 2005

Total points: 50 — Time limit: 1 hour

No calculators, books, notes, or other aids are permitted. You must **show your work** and **justify your steps** to receive full credit.

1. (5 points) Short Answer:

   (a) Complete the following sentence with the definition of continuity: We say that $f(x)$ is continuous at $x = a$ if . . .

   Solution: . . . $\lim_{x\to a} f(x) = f(a)$. OR: . . . $f(a)$ is defined, $\lim_{x\to a} f(x)$ exists, and the two are equal.

   (b) Explain why the following reasoning is incorrect:

   > Let $f(x) = [\![x]\!]$. We know that $f(1.5) = 1$ and $f(2.5) = 2$. Also, 1.3 is between 1 and 2, so according to the Intermediate Value Theorem, there must be a point c between 1.5 and 2.5 where $f(c) = 1.3$.

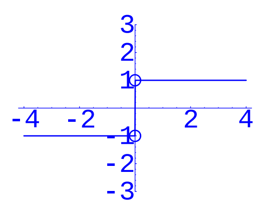
   Solution: The Intermediate Value Theorem only applies to continuous functions. The greatest integer function has a discontinuity between 1.5 and 2.5 (namely, at $x = 2$), so it doesn't apply.

   (c) True or False: If $f(x)$ is defined at $x = a$, then you can compute $\lim_{x\to a} f(x)$ by plugging in.

   Solution: FALSE. For example, $f(x) = [\![x]\!]$ is defined at $x = 3$, but $\lim_{x\to 3} f(x)$ doesn't exist.

   (d) Give an example (by sketching a graph) of an instance in which both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, but $\lim_{x\to a} f(x)$ does not.

   Solution: In the following function, $\lim_{x\to 0^+} f(x) = 1$, while $\lim_{x\to 0^-} f(x) = -1$. Since the one-sided limits are different, the total (two-sided) limit does not exist.



   (e) True or False: A function may have infinitely many horizontal asymptotes, but can only have at most two vertical asymptotes.

   Solution: FALSE. It's the other way around.

2. (16 points) Analyze the following limits:

   (a) $\lim_{x\to 2} \dfrac{x^2 - 5x + 6}{x - 2}$

   Solution:
   $$= \lim_{x\to 2} \frac{(x-2)(x-3)}{x-2} = \lim_{x\to 2}(x-3) = 2 - 3 = -1.$$

(b) $\lim_{x\to 3^-} \dfrac{x^2+2}{x(x-3)}$

Solution: If you try to plug in $x=3$, you get something of the form $\frac{\text{nonzero}}{0}$, so the limit must be either ∞ or $-\infty$. If x is close to 3 and slightly less than it, then:

$$\frac{x^2+2}{x(x-3)} \qquad \text{is} \qquad \frac{\text{positive}}{(\text{positive})(\text{negative})},$$

so the whole fraction is negative. So $\lim_{x\to 3^-} \dfrac{x^2+2}{x(x-3)} = -\infty$.

(c) $\lim_{x\to\infty} \dfrac{x^3+1}{\sqrt{9x^6-1}+x}$

Solution:

$$= \lim_{x\to\infty} \frac{x^3+1}{\sqrt{9x^6-1}+x} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{1+1/x^3}{\sqrt{9x^6-1}\sqrt{1/x^6}+x(1/x^3)}$$

$$= \lim_{x\to\infty} \frac{1+1/x^3}{\sqrt{9-1/x^6}+1/x^2} = \frac{1+0}{\sqrt{9-0}+0} = \frac{1}{3}.$$

(d) $\lim_{x\to 3^-} [\![-x]\!]$

Solution: If x is close to 3 and slightly less than it (e.g., 2.8, 2.95) then $-x$ is close to 3 and slightly *greater* than it (e.g. -2.8, -2.95). The greatest integer function applied to such numbers always gives -3 (that is, $[\![-2.8]\!] = [\![-2.95]\!] = -3$). So $\lim_{x\to 3^-} [\![-x]\!] = -3$.

3. (4 points) For what value of a is the following function continuous?

$$f(x) = \begin{cases} \sqrt{3x^2+a^2} & \text{if } x < 2 \\ 3x+a & \text{if } x \geq 2 \end{cases}$$

Solution: For f to be continuous, its right- and left-hand limits have to be equal.

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} \sqrt{3x^2+a^2} \\ &= \sqrt{3(2)^2+a^2} \\ &= \sqrt{12+a^2} \end{aligned} \qquad\qquad \begin{aligned} \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (3x+a) \\ &= 3(2)+a \\ &= 6+a \end{aligned}$$

Setting these equal, we have

$$\begin{aligned} \sqrt{12+a^2} &= 6+a \\ 12+a^2 &= (6+a)^2 = 36+12a+a^2 \\ -24 &= 12a \\ a &= -2 \end{aligned}$$

4. (5 points) Find the horizontal asymptote(s) of the function $g(t) = \dfrac{3x^2+5}{x^2-9}$.

Solution: To find the horizontal asymptotes, we calculate:

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2+5}{x^2-9} &= \lim_{x\to\infty} \frac{3x^2+5}{x^2-9} \cdot \frac{1/x^2}{1/x^2} \\ &= \lim_{x\to\infty} \frac{3+5/x^2}{1-9/x^2} = \frac{3+0}{1-0} = 3. \end{aligned}$$

We get the same answer if we compute $\lim\limits_{x\to-\infty} \dfrac{3x^2+5}{x^2-9}$, so $y=3$ is the only horizontal asymptote.

5. (10 points) On the planet Schmearth (it's like Earth, but different), the laws of physics are not the same as here. For example, if a schmearthling drops a ball from a height of 18 meters, then the height of the ball after t seconds is given by
$$h(t) = \frac{18}{t+1}.$$

   (a) What is the average velocity of the ball during the first five seconds (that is, from $t=0$ to $t=5$)?

   Solution:
   $$\frac{h(0)-h(5)}{0-5} = \frac{\frac{18}{0+1} - \frac{18}{5+1}}{-5} = \frac{18-3}{-5} = \frac{15}{-5} = -3 \text{ m/s}$$

   (b) What is the instantaneous velocity of the ball at $t=2$?

   Solution: (In this calculation, I used $\lim\limits_{k\to 0}$ instead of $\lim\limits_{h\to 0}$ just to avoid confusion between h the function and h the limit variable.)
   $$\begin{aligned}\lim_{k\to 0} \frac{h(2+k)-h(2)}{k} &= \lim_{k\to 0} \frac{\frac{18}{(2+k)+1} - \frac{18}{2+1}}{k} = \lim_{k\to 0} \frac{\frac{18}{3+k} - 6}{k} \cdot \frac{3+k}{3+k} \\ &= \lim_{k\to 0} \frac{18-6(3+k)}{k(3+k)} = \lim_{k\to 0} \frac{18-18-6k}{k(3+k)} = \lim_{k\to 0} \frac{-6k}{k(3+k)} \\ &= \lim_{k\to 0} \frac{-6}{3+k} = -2 \text{ m/s}\end{aligned}$$

   (c) After how many seconds does the ball reach the ground? (This is sort of a trick question.)

   Solution: Never. When the ball reaches the ground, its height above the ground should be 0, but if we set the height equal to 0,
   $$\frac{18}{t+1} = 0,$$
   we see that there are no solutions.

6. (6 points) For each of the following functions, determine the type of discontinuity (removable, jump, or infinite) at the given point.

   (a) $f(x) = \dfrac{x(x-3)}{(x+2)(x-3)}, \quad x = 3$

   Solution: Removable. Note that
   $$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x(x-3)}{(x+2)(x-3)} = \lim_{x\to 3} \frac{x}{x+2} = \frac{3}{5}.$$
   Since the limit exists at $x=3$ but the function isn't defined, the discontinuity is removable.

   (b) $f(x) = \dfrac{|x+2|}{x+2}, \quad x = -2$

   Solution: Jump. We have
   $$\lim_{x\to -2^-} \frac{|x+2|}{x+2} = -1 \qquad \text{and} \qquad \lim_{x\to -2^+} \frac{|x+2|}{x+2} = 1.$$
   Since the left- and right-hand limits exist but are different, it's a jump discontinuity.

7. (4 points) Find the domain of $k(x) = 2\sqrt{x+4}$.

Solution: This function is defined as long as the quantity under the radical is not negative: that is, $x+4 \geq 0$, or $x \geq -4$. So the domain is $[-4, \infty)$.