Exam 2 Solutions

October 4, 2005

Total points: 50 | Time limit: 1 hour

No calculators, books, notes, or other aids are permitted. You must **show your work** and **justify your steps** to receive full credit.

1. (4 points) Short answer. **Warning:** At least one of the questions below is a trick question. If you think a question has no meaningful answer, write the words "TRICK QUESTION" as your answer.

   (a) State *both* limit definitions of $f'(x)$.

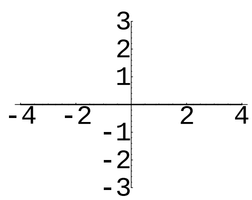
   Solution: $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ and $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$

   (b) Suppose the graph of $h(x)$ passes through the point $(5,3)$, and the tangent line to the graph at this point is given by $y=-3x+18$. What is $h'(5)$?

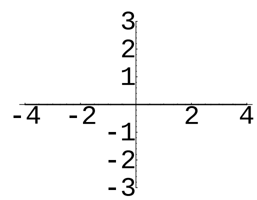
   Solution: -3, which is the slope of the tangent line.

   (c) Give examples (by sketching graphs) of a function that is:

   (i) continuous but not differentiable at $x=0$

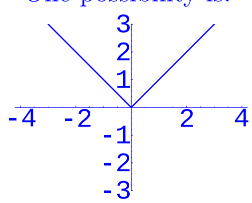


   (ii) differentiable but not continuous at $x=0$



   Solution:

   One possibility is:



   TRICK QUESTION. If a function is not continuous at $x=0$, it can't be differentiable there either.

2. (3 points) Evaluate the following limit: $\lim\limits_{t\to 0} \dfrac{5t}{\tan t}$

   Solution:

$$= \lim_{t\to 0} \frac{5t}{\frac{\sin t}{\cos t}} = \lim_{t\to 0} \frac{5t}{\frac{\sin t}{\cos t}} \cdot \frac{\frac{\cos t}{t}}{\frac{\cos t}{t}}$$

$$= \lim_{t\to 0} \frac{5\cos t}{\frac{\sin t}{t}} = \frac{\lim_{t\to 0} 5\cos t}{\lim_{t\to 0} \frac{\sin t}{t}}$$

$$= \frac{5\cdot 1}{1} = 5.$$

3. (5 points) If $f(t) = t^2 + \dfrac{1}{t^2} + 10^t + 5\cos t$, find $f'(t)$.

Solution:

$$\begin{aligned} f'(t) &= \frac{d}{dt}(t^2) + t^{-2} + 10^t + 5\cos t) \\ &= 2t^{2-1} + (-2)t^{-2-1} + 10^t \ln 10 + 5(-\sin t) \\ &= 2t - 2t^{-3} + 10^t \ln 10 - 5\sin t \end{aligned}$$

4. (5 points each) Choose **four** out of the following **five** functions, and find their derivatives. **You must clearly indicate any place where you use the Product Rule or Quotient Rule.**

(a) $q(s) = s^e e^s$

Solution:

$$\begin{aligned} q'(s) &= s^e \frac{d}{ds} e^s + e^s \frac{d}{ds} s^e && \text{product rule} \\ &= s^e e^s + e^s \frac{d}{ds} s^e && \text{exponential rule} \\ &= s^e e^s + e^s \cdot e s^{e-1} && \text{power rule} \\ &= s^e e^s + e^{s+1} s^{e-1} \end{aligned}$$

(b) $R(t) = (t + e^t)(3 - \sqrt{t})$

Solution:

$$\begin{aligned} R(t) &= (t + e^t)(3 - t^{1/2}) \\ R'(t) &= (t + e^t)\frac{d}{dt}(3 - t^{1/2}) + (3 - t^{1/2})\frac{d}{dt}(t + e^t) && \text{product rule} \\ &= (t + e^t)(0 - (1/2)t^{-1/2}) + (3 - t^{1/2})(1 + e^t) \\ &= -\frac{t + e^t}{2t^{1/2}} + (3 - \sqrt{t})(1 + e^t) \end{aligned}$$

(c) $g(x) = \dfrac{1}{2 + 3e^x}$

Solution:

$$\begin{aligned} g'(x) &= \frac{(2 + 3e^x)\frac{d}{dx}1 - 1\frac{d}{dx}(2 + 3e^x)}{(2 + 3e^x)^2} && \text{quotient rule} \\ &= \frac{(2 + 3e^x) \cdot 0 - 1 \cdot (0 + 3e^x)}{(2 + 3e^x)^2} \\ &= -\frac{3e^x}{(2 + 3e^x)^2} \end{aligned}$$

(d) $h(x) = \dfrac{1 + \sin x}{x + \cos x}$

Solution:

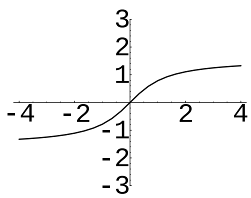
$$\begin{aligned} h'(x) &= \frac{(x + \cos x)\frac{d}{dx}(1 + \sin x) - (1 + \sin x)\frac{d}{dx}(x + \cos x)}{(x + \cos x)^2} && \text{quotient rule} \\ &= \frac{(x + \cos x)(0 + \cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} \\ &= \frac{x\cos x + \cos^2 x - (1 - \sin^2 x)}{(x + \cos x)^2} \\ &= \frac{x\cos x + (\cos^2 x + \sin^2 x) - 1}{(x + \cos x)^2} && \text{use } \cos^2 x + \sin^2 x = 1 \\ &= \frac{x\cos x}{(x + \cos x)^2} \end{aligned}$$

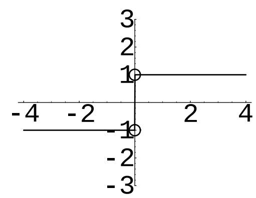
(e) $k(x) = xe^x \sin x$

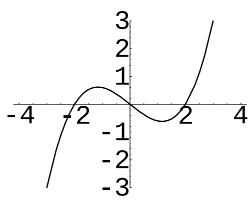
Solution:

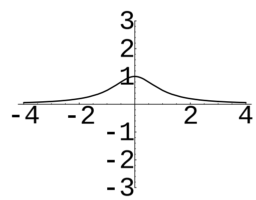
$$\begin{aligned} k'(x) &= xe^x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} xe^x && \text{product rule} \\ &= xe^x \cos x + \sin x \,(x \frac{d}{dx} e^x + e^x \frac{d}{dx} x) && \text{product rule again} \\ &= xe^x \cos x + \sin x \,(xe^x + e^x \cdot 1) \\ &= xe^x \cos x + xe^x \sin x + e^x \sin x. \end{aligned}$$

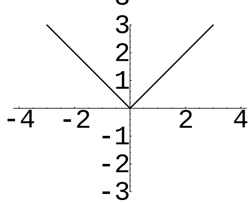
5. (6 points) Match the graph of each function in (A)–(D) with the graph of its derivative in (1)–(4).

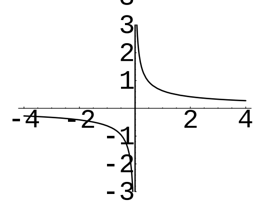
A. 

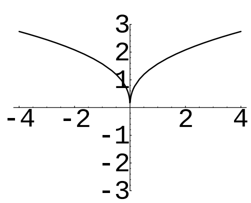
1. 

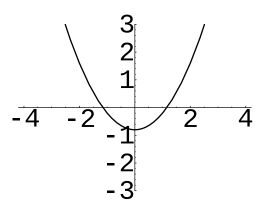
B. 

2. 

C. 

3. 

D. 

4. 

Solution: A–2, B–4, C–1, D–3.

6. (4 points) Use the quotient rule, the derivatives of $\sin x$ and $\cos x$, and trigonometric identities to find the derivative of

$$f(x) = \cot x.$$

You will not receive credit for simply writing down the derivative of $\cot x$.

Solution:

$$
\begin{aligned}
\frac{d}{dx}\cot x &= \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x \frac{d}{dx}\cos x - \cos x \frac{d}{dx}\sin x}{\sin^2 x} \\
&= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

7. (8 points) This morning, as I got my bicycle out, I noticed that the front tire was low on air. The pressure was 40 psi (pounds per square inch), but it should have been closer to 70 psi. As I pumped up the tire, the pressure after t minutes was given by

$$P(t) = 120 - \frac{80}{2^t}.$$

You may use the approximation $\ln 2 \approx \frac{7}{10}$ in this problem.

(a) Find $P'(t)$.

Solution:

$$
\begin{aligned}
P'(t) &= \frac{d}{dt}\left(120 - \frac{80}{2^t}\right) \\
&= 0 - 80\frac{d}{dt}\frac{1}{2^t} \\
&= -80\frac{2^t \frac{d}{dt}1 - 1\frac{d}{dt}2^t}{(2^t)^2} && \text{quotient rule} \\
&= -80\frac{2^t \cdot 0 - 2^t \ln 2}{(2^t)^2} = 80\frac{2^t \ln 2}{(2^t)^2} \\
&= \frac{80 \ln 2}{2^t} \approx \frac{80 \cdot 0.7}{2^t} = \frac{56}{2^t}
\end{aligned}
$$

(b) What is the (instantaneous) rate of change of the pressure right when I start pumping (that is, at $t = 0$)? What is the rate of change after 1 minute? Be sure to include units with your answers.

Solution:

Rate of change at $t = 0$: $P'(0) = \frac{56}{2^0} = 56$ psi/min

Rate of change at $t = 1$: $P'(1) = \frac{56}{2^1} = 28$ psi/min

(c) Should I even still be pumping after 1 minute? (In other words, does the tire have enough air by then?)

Solution: $P(1) = 120 - \frac{80}{2^1} = 120 - 40 = 80$ psi. The tire has enough air by then, so No, I do not need to keep pumping after 1 minute.