Math 1550 | Fall 2005

Section 31 | P. Achar

Exam 3 Solutions

October 27, 2005

Total points: 50 | Time limit: 1 hour

No calculators, books, notes, or other aids are permitted. You must **show your work** and **justify your steps** to receive full credit.

1. (5 points) Short answer:

   (a) Suppose $h(x) = f(g(x))$. Given the following information, find $h'(3)$.

   $f(2) = 1 \quad f(3) = 4 \quad g(3) = 2 \quad g(4) = 3$

   $f'(2) = 7 \quad f'(3) = 3 \quad g'(3) = 5 \quad g'(4) = -1$

   $h'(3) =$

   Solution: The chain rule says $h'(x) = f'(g(x))g'(x)$, so

   $$h'(3) = f'(g(3))g'(3) = f'(2)f'(3) = 7 \cdot 5 = 35.$$

   (b) What is the point of logarithmic differentiation? (In other words, what can one do with logarithmic differentiation that one can't do otherwise?)

   Solution: Functions of the form $f(x)^{g(x)}$ (like $(\sin x)^x$) don't fit any of the differentiation rules that we have. To find the derivative of such a function, we must use logarithmic differentiation.

   (c) Suppose you are driving an automobile. Let $D(x)$ be the distance you've traveled after using up x gallons of gasoline. What is the meaning of $D'(x)$? What are its units?

   Solution: $D'(x)$ is your "instantaneous mileage"; its units are miles per gallon.

(4 points each) In problems 2–6, find dy/dx. Clearly indicate any place where you use the **chain rule** or **logarithmic differentiation**.

2. $y = \dfrac{e^{2t}}{e^t + e^{-t}}$

   Solution:

   $$\begin{aligned}
   \frac{dy}{dx} &= \frac{(e^t + e^{-t})\frac{d}{dt}e^{2t} - e^{2t}\frac{d}{dt}(e^t + e^{-t})}{(e^t + e^{-t})^2} && \text{quotient rule} \\
   &= \frac{(e^t + e^{-t})e^{2t} \cdot 2 - e^{2t}(e^t + e^{-t} \cdot (-1))}{(e^t + e^{-t})^2} && \text{chain rule: } \tfrac{d}{dt}e^{2t} = e^{2t} \cdot 2,\ \tfrac{d}{dt}e^{2t} = e^{-t} \cdot (-1) \\
   &= \frac{2e^{2t}(e^t + e^{-t}) - e^{2t}(e^t - e^{-t})}{(e^t + e^{-t})^2} \\
   &= \frac{2e^{3t} + 2e^t - e^{3t} + e^t}{(e^t + e^{-t})^2} \\
   &= \frac{e^{3t} + 3e^t}{(e^t + e^{-t})^2}
   \end{aligned}$$

3. $y = (\sin x)^x$.

   Solution:

$$\ln y = \ln(\sin x)^x = x \ln \sin x \qquad \text{apply ln to both sides}$$

$$\frac{1}{y}y' = x\frac{d}{dx}\ln\sin x + \ln\sin x \cdot 1 \qquad \text{differentiate: chain rule for } \ln y\text{; product rule for } x\ln\sin x$$

$$= x \cdot \frac{1}{\sin x}\cos x + \ln \sin x \qquad \text{chain rule for } \ln\sin x\text{: } \begin{array}{ll} f(u)=\ln u & g(x)=\sin x \\ f'(u)=\frac{1}{u} & g'(x)=\cos x \end{array}$$

$$y' = y\left(\frac{x\cos x}{\sin x} + \ln\sin x\right)$$

$$y' = (\sin x)^x\left(\frac{x\cos x}{\sin x} + \ln\sin x\right)$$

4. $y = \log_8(\cos x^2)$

Solution:

$$y' = \frac{1}{\cos x^2 \ln 8}\frac{d}{dx}\cos x^2 \qquad \text{chain rule: } \begin{array}{ll} f(u)=\log_8 u & g(x)=\cos x^2 \\ f'(u)=\frac{1}{u\ln 8} & g'(x)=??? \end{array}$$

$$= \frac{1}{\cos x^2 \ln 8}(-\sin x^2)\cdot 2x \qquad \text{chain rule for } g(x)\text{: } \begin{array}{ll} h(v)=\cos v & j(x)=x^2 \\ h'(v)=-\sin v & j'(x)=2x \end{array}$$

$$= \frac{-2x\sin x^2}{\cos x^2 \ln 8} = -\frac{2}{\ln 8}x\tan x^2.$$

5. $y = \sqrt{x+\sqrt{x}}$

Solution:

$$\frac{dy}{dx} = \frac{1}{2}(x+\sqrt{x})^{-1/2}\cdot\left(1+\frac{1}{2}x^{-1/2}\right) \qquad \text{chain rule: } \begin{array}{ll} f(u)=\sqrt{u} & g(x)=x+\sqrt{x} \\ f'(u)=\frac{1}{2}u^{-1/2} & g'(x)=1+\frac{1}{2}x^{-1/2} \end{array}$$

6. $y = \sqrt{\arctan x}$

Solution:

$$y' = \frac{1}{2}(\arctan x)^{-1/2}\cdot\frac{1}{1+x^2} \qquad \text{chain rule: } \begin{array}{ll} f(u)=\sqrt{u} & g(x)=\arctan x \\ f'(u)=\frac{1}{2}u^{-1/2} & g'(x)=\frac{1}{1+x^2} \end{array}$$

$$= \frac{1}{2(1+x^2)\sqrt{\arctan x}}.$$

7. (5 points) Let $y^3 + x^2y = e^{-x}$.

   (a) Find dy/dx by implicit differentiation.

   Solution:

$$3y^2\frac{dy}{dx} + \left(x^2\frac{dy}{dx} + 2xy\right) = e^{-x}\cdot(-1) \qquad \text{chain rule for } y^3 \text{ and } e^{-x}\text{, product rule for } x^2y$$

$$(3y^2+x^2)\frac{dy}{dx} = -e^{-x} - 2xy$$

$$\frac{dy}{dx} = -\frac{e^{-x}+2xy}{3y^2+x^2}$$

   (b) Find the slope of the tangent line to the curve at the point $(0,1)$.

   Solution:

$$\frac{dy}{dx}(0,1) = -\frac{e^{-0}+2\cdot 0\cdot 1}{3\cdot 1^2+0^2} = -\frac{1+0}{3+0} = -\frac{1}{3}.$$

8. (5 points) Let $y = e^{2x}$. Find d^2y/dx^2.

Solution:

$$y' = e^{2x}\cdot 2 = 2e^{2x} \qquad \text{chain rule}$$

$$y'' = 2e^{2x}\cdot 2 = 4e^{2x} \qquad \text{chain rule again}$$

9. (3 points) Find the first four derivatives of $-\cos x$.

Solution:

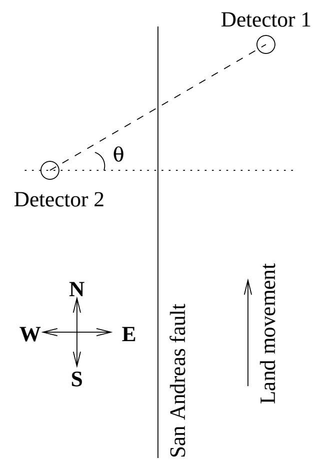
$$\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x \qquad \frac{d^3}{dx^3}(-\cos x) = \frac{d}{dx}\cos x = -\sin x$$
$$\frac{d^2}{dx^2}(-\cos x) = \frac{d}{dx}\sin x = \cos x \qquad \frac{d^4}{dx^4}(-\cos x) = \frac{d}{dx}(-\sin x) = -\cos x$$

10. (2 points) Find $\frac{d^{133}}{dx^{133}}(-\cos x)$.

Solution: The pattern of derivatives of $-\cos x$ repeats after every four derivatives, so the 4th, 8th, 12th, *etc.* derivatives are all equal to $-\cos x$. 132 is divisible by 4, so the 132nd derivative is $-\cos x$, so

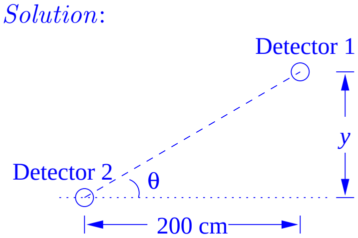
$$\frac{d^{133}}{dx^{133}}(-\cos x) = \frac{d}{dx}(-\cos x) = \sin x.$$

11. (10 points) The San Andreas fault in California is a crack in the surface of the earth that runs north-south. The land mass to the east of the fault is slowly moving northward. To measure the movement of the land along this fault, a seismologist sets up two detectors, one 100 cm to the east of the fault line, and the other 100 cm to the west. At the beginning, Detector 1 was due east of Detector 2, but over time, Detector 1 has moved north. The detectors report the angle θ by which Detector 1 is off from being due east of Detector 2. If the angle is increasing by $\frac{1}{200}$ radian each year, how fast is the eastern landmass moving north when $\theta = \pi/3$? (*Note*: This is roughly how seismologists measure movement along fault lines in real life.)



(a) List all the variables you will use, and label them on the picture.

Solution:



Variables:

θ = angle between Detector 1 and due east

y = distance north that Detector 1 has moved

t = time in years

(b) List the all the relationships between variables that you can find.

Solution: $\tan\theta = \frac{y}{200}$

(c) List known information about derivatives, and indicate what derivative you must find to answer the question.

Solution: Known information: $\frac{d\theta}{dr} = \frac{1}{200}$ rad/yr

Looking for: $\frac{dy}{dt}$ when $\theta = \pi/3$.

(d) Finish solving the problem.

Solution:

$$\begin{aligned}
\tan\theta &= \frac{y}{200} \\
\sec^2\theta\,\frac{d\theta}{dt} &= \frac{1}{200}\frac{dy}{dt} && \text{implicit differentiation} \\
\frac{1}{\cos^2\theta}\frac{d\theta}{dt} &= \frac{1}{200}\frac{dy}{dt} \\
\frac{1}{\cos^2\pi/3}\frac{1}{200} &= \frac{1}{200}\frac{dy}{dt} && \text{plug in } \frac{d\theta}{dt} = \frac{1}{200},\ \theta = \pi/3 \\
\frac{1}{(1/2)^2} &= \frac{dy}{dt} && \text{cancel } 1/200\text{, and use } \cos\pi/3 = 1/2 \\
\frac{dy}{dt} &= 4
\end{aligned}$$

Thus, the eastern landmass is moving north at a rate of 4 cm/yr.