Math 1550 Fall 2005
Section 31 P. Achar

Exam 4 Solutions

November 17, 2005

Total points: 50 Time limit: 1 hour

No calculators, books, notes, or other aids are permitted. You must **show your work** and **justify your steps** to receive full credit.

1. (5 points) Short answer:

   (a) State the definition of *critical point*.

   Solution: a point where the derivative is 0 or undefined

   (b) Suppose $f'(3) = 0$ and $f''(3) = 2$. According to the second derivative test, what can you say about $f(x)$ at $x = 3$?

   Solution: The second derivative test says that $f(x)$ has a local minimum at $x = 3$.

   (c) True or false: If $f'(4) = 0$, then $f(x)$ *must* have either a maximum or a minimum at $x = 4$.

   Solution: FALSE. For example, the function $f(x) = (x-4)^3$ has the property that $f'(4) = 0$, but it has neither a local minimum nor a local maximum at $x = 4$.

2. (5 points) Evaluate the following limit:
$$\lim_{x\to\infty} \frac{\ln \ln x}{x}$$

   Solution:
$$\begin{aligned} &\lim_{x\to\infty} \frac{\ln \ln x}{x} && \text{Form: } \frac{\infty}{\infty} \\ &= \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot 1x}{1} && \text{L'Hôpital's Rule} \\ &= \lim_{x\to\infty} \frac{1}{x \ln x} && \text{Form: } \frac{1}{\infty} \\ &= 0. \end{aligned}$$

   Answer **two** of the **three** questions on this page. (5 points each)

3. Evaluate the following limit: $\lim_{t\to 0} \dfrac{e^{3t}-1}{t}$

   Solution:
$$\begin{aligned} &\lim_{t\to 0} \frac{e^{3t}-1}{t} && \text{Form: } \frac{e^0-1}{0} = \frac{1-1}{0} = \frac{0}{0} \\ &= \lim_{t\to 0} \frac{e^{3t}\cdot 3}{1} && \text{L'Hôpital's Rule} \\ &= \lim_{t\to 0} 3e^{3t} && \text{now, can evaluate by plugging in} \\ &= 3. \end{aligned}$$

4. Let $r(x) = x - \arctan x$. Show that $r(x)$ has no local minima or maxima.

Solution: First, we find the critical points:

$$\begin{aligned} r'(x) = 1 - \frac{1}{1+x^2} &= 0 \\ 1 &= \frac{1}{1+x^2} \\ 1 + x^2 &= 1 \\ x^2 &= 0 \\ x &= 0 \end{aligned}$$

Next, we use the first derivative test:

$$\begin{array}{lcc} r'(x): & + & + \\ \hline & \multicolumn{2}{c}{0} \end{array}$$

So $x = 0$ is neither a local minimum nor a local maximum.

5. The function $m(x) = 3x^4 - 20x^3$ has critical points at $x = 0$ and $x = 5$. Using *only the second derivative test*, determine whatever you can about whether $f(x)$ has minima or maxima at these points.

   Solution: First, we compute the second derivative:

   $$\begin{aligned} m'(x) &= 12x^3 - 60x^2 \\ m''(x) &= 36x^2 - 120x \end{aligned}$$

   Next, we plug in $x = 0$ and $x = 5$:

   $m''(0) = 36 \cdot 0 - 120 \cdot 0 = 0$ $\Rightarrow$ The second derivative test gives no information.
   $m''(5) = 36 \cdot 25 - 120 \cdot 5 = 900 - 600 = 300 > 0$ $\Rightarrow$ $m(x)$ has a local minimum at $x = 5$.

Answer **one** of the **two** questions on this page. (8 points each)

6. Consider the function $g(t) = t^4 - 2t^2 + 3$.

   (a) Find its critical points.

   Solution:
   $$\begin{aligned} g'(t) = 4t^3 - 4t &= 0 \\ 4t(t^2 - 1) &= 0 \\ t &= 0,\ 1,\ -1 \end{aligned}$$

   (b) Find its absolute maximum in the interval $[1, 3]$.

   Solution: We have to check the value of the function at the critical points and endpoints of the interval. However, the critical points 0 and -1 are not in the interval $[1, 3]$, so they are irrelavant to the problem.

   Critical point:

   $t = 1$ $\quad$ $g(1) = 1^4 - 2 \cdot 1^2 + 3 = 2$

   Endpoints:

   $t = 1$ $\quad$ $g(1) = 2$

   $t = 3$ $\quad$ $g(3) = 3^4 - 2 \cdot 3^2 + 3 = 66$

   The absolute maximum occurs at $t = 3$, and $g(3) = 66$.

7. Consider the function $f(x) = x^3 + 2x - 2$.

   (a) Find the slope of the secant line to the graph of this function from $x = 0$ to $x = 2$.

   Solution:

$$\begin{aligned}\frac{f(2)-f(0)}{2-0} &= \frac{(2^3+2\cdot 2-2)-(0^3+2\cdot 0-2)}{2} \\ &= \frac{10-(-2)}{2} = 6.\end{aligned}$$

   (b) Find all points c in the interval $[0, 2]$ that satisfy the conclusion of the Mean Value Theorem.

   Solution:

$$\begin{aligned}f'(c) = 3c^2 + 2 &= 6 \\ 3c^2 &= 4 \\ c &= \pm\sqrt{4/3} = \pm 2/\sqrt{3}\end{aligned}$$

   But $-2/\sqrt{3}$ is not in the interval $[0, 2]$, so the only such point is $c = 2/\sqrt{3}$.

8. (12 points) Consider the function $f(x) = x^4 - 8x^2 + 5$.

   (a) Find its critical points.

   Solution:

$$\begin{aligned}f'(x) = 4x^3 - 16x &= 0 \\ 4x(x^2-4) &= 0 \\ x = 0 \quad &\text{or} \quad x^2 - 4 = 0 \\ x = 0 \quad &\text{or} \quad x = \pm 2\end{aligned}$$

   (b) Using the first derivative test, determine whether each critical point is a local maximum, a local minimum, or neither.

   Solution: Plugging in various points, we find:

$$\begin{array}{ccccccccc} f'(x): & & - & & + & & - & & + \\ \hline & & & -2 & & 0 & & 2 & \end{array}$$

   So $f(x)$ has minima at $x = -2$ and $x = 2$, and a maximum at $x = 0$.

   (c) Find the inflection points of $f(x)$.

   Solution: We have

$$\begin{aligned}f''(x) = 12x^2 - 16 &= 0 \\ 12x^2 &= 16 \\ x^2 &= 4/3 \\ x &= \pm\sqrt{4/3} = \pm 2/\sqrt{3}.\end{aligned}$$

   Next,

$$\begin{array}{cccccc} f''(x): & + & & - & & + \\ \hline & & -2/\sqrt{3} & & 2/\sqrt{3} & \end{array}$$

   So both points $x = \pm 2/\sqrt{3}$ are inflection points.

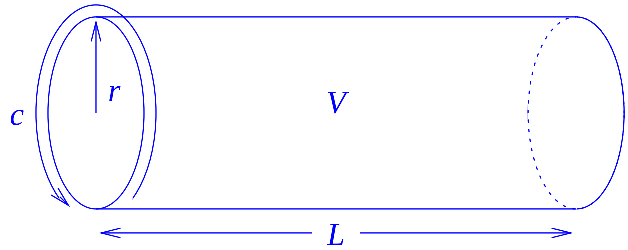
(d) On what intervals is $f(x)$ concave up? On what intervals is it concave down?

Solution: From the above chart of the second derivative, we see that $f(x)$ is concave up on $(-\infty, -2/\sqrt{3})$ and $(2/\sqrt{3}, \infty)$ and concave down on $(-2/\sqrt{3}, 2/\sqrt{3})$.

9. (10 points) Suppose you want to send a cylindrical package (like a poster tube) through the mail. The U.S. Postal Service imposes a size restriction: they won't accept such a package if the sum of its circumference and its length exceeds 45π inches. What is the volume of the largest cylindrical package you can mail?

(a) Sketch a picture, and list and label all the variables you will use.

Solution:



r = radius
c = circumference
L = length
V = volume

(b) List all the relationships between the variables you can find.

Solution:

$$c = 2\pi r$$
$$c + L = 45\pi$$
$$V = \pi r^2 L$$

(c) Indicate which variable you must minimize or maximize, and express that variable as a function of one other variable.

Solution: We want to maximize V. Solving for L and substituting into V, we have

$$\begin{aligned} L &= 45\pi - c = 45\pi - 2\pi r \\ V &= \pi r^2 L = \pi r^2(45\pi - 2\pi r) \\ &= 45\pi^2 r^2 - 2\pi^2 r^3. \end{aligned}$$

(d) Finish solving the problem. You must check that you found the correct type of extremum, using the first or second derivative test, in order to receive full credit.

Solution: We have

$$\begin{aligned} V' = 90\pi^2 r - 6\pi^2 r^2 &= 0 \\ 6\pi^2 r(15 - r) &= 0 \end{aligned}$$

so the critical points are $r = 0$ and $r = 15$. First derivative test:

V' : $-$ (left of 0), $+$ (between 0 and 15), $-$ (right of 15)

0 15

so there is a maximum at $r = 15$. Thus, the maximum possible volume is

$$\begin{aligned} V(15) &= 45\pi^2 \cdot 15^2 - 2\pi^2 \cdot 15^3 \\ &= 3 \cdot 15\pi^2 \cdot 15^2 - 2\pi^2 \cdot 15^3 = 3\pi^2 \cdot 15^3 - 2\pi^2 \cdot 15^3 \\ &= 15^3\pi^2. \end{aligned}$$