Problem Set 1a

Due: September 6, 2005

1. Let W be a reflection group acting a real vector space V. Show that V admits an inner product (*i.e.*, a positive-definite symmetric bilinear form) $\langle , \rangle$ with respect to which W consists of orthogonal transformations.

2. Let k be an arbitrary field, and V a vector space over k. Let $s : V \to V$ be a linear transformation that fixes some hyperplane H pointwise.
   (a) Suppose $s^2 = 1$. Under what conditions does it follow that s is a reflection? (*Hint:* Consider the characteristic of k.)
   (b) More generally, s is called a *pseudoreflection* if, in addition to fixing H pointwise, s has a one-dimensional eigenspace with eigenvalue a root of unity. If some power of s is the identity map on V, then under what conditions does it follow that s is a pseudoreflection?

3. Let V be a vector space. Recall that Humphreys' axioms for a root system $\Phi \subset V$ are:
   (R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;
   (R2) $s_\alpha \Phi = \Phi$.

   Consider the following axiom

   (R2′) For each $\alpha \in \Phi$, there exists an element $\alpha^\vee \in V^*$ such that $\alpha^\vee(\alpha) = 2$, and such that $p_\alpha(\Phi) = \Phi$, where $p_\alpha : V \to V$ is the map defined by $p(x) = x - \alpha^\vee(x)\alpha$.

   (a) Show that a set Φ satisfies (R1) and (R2) if and only if it satisfies (R1) and (R2′). Thus, (R1) and (R2′) together provide an alternate definition of root system, one that does not require V to be endowed with an inner product beforehand.
   (b) Assume now that Φ is a root system that spans V. Show that the elements $\alpha^\vee \in V^*$ are uniquely determined, and that the entire set $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ is itself a root system. What can you say about the reflection group associated to $\Phi^\vee$? What can you say about $(\Phi^\vee)^\vee$?

4. (a) Prove that any reflection group generated by two reflections is isomorphic to a dihedral group.
   (b) (*cf.* Humphreys, Exercise 1.3.1) Prove that any reflection group acting on a 2-dimensional vector space can be generated by two reflections.

5. (Humphreys, Exercise 1.3.2) Find simple systems for each of the groups $\mathcal{S}_n$, B_n, D_n, and $\mathcal{D}_m$.