Problem Set 2a

Due: September 27, 2005

1. Recall that a parabolic subgroup of a Coxeter group is a subgroup generated by a subset of the Coxeter generators. Similarly, a subgroup of a reflection group is said to be *parabolic* if there exists a choice of simple system with respect to which it is generated by a subset of the simple reflections.

   Let W be a reflection group acting on the real vector space V. Prove that a subgroup $W' \subset W$ is parabolic if and only if there is a subspace $V' \subset V$ such that W' is the pointwise stabilizer of V' in W. (That is, $W' = \{w \in W \mid wv = v \text{ for all } v \in V'\}$.)

2. A (true) root system Φ (as opposed to what I have called a "pseudo-root system") is called *crystallographic* if $2\langle\alpha, \beta\rangle / \langle\alpha, \alpha\rangle \in \mathbb{Z}$ for all roots $\alpha, \beta \in \Phi$. A reflection group is called *crystallographic* if it has a crystallographic root system.

   (a) Find two inequivalent (*i.e.*, not scalar multiples of one another) crystallographic root systems for the group B_n. This proves not only that B_n is a crystallographic reflection group, but also A_n and D_n, since those are subgroups of B_n. (*Historical note:* Traditionally, the name "B_n" is attached to just one of these root systems (namely, the one in which the roots for transpositions are longer than the roots for sign changes). The other root system is called C_n. Indeed, the names "A_n" and "D_n" should also properly be thought of as names of root systems rather than reflection groups, which leads in to the next question . . .)

   (b) Show that all root systems for A_n are scalar multiples of one another. Then, do the same thing for D_n.

   (c) Which of the dihedral groups $I_2(m)$ are crystallographic?

   (d) (Answer this question after we complete the classification of reflection groups in class.) Which of the irreducible reflection groups are crystallographic?

   (The topics in this question come up in various other contexts. For instance, the isomorphism classes of simply-connected compact Lie groups are in one-to-one correspondence with equivalence classes of crystallographic root systems. The same is true of simple complex Lie algebras.)

3. Humphreys defines "crystallographic reflection groups" differently. His definition essentially says that W is crystallographic if, by choosing a suitable basis for V (and thus identifying $GL(V) \simeq GL(n, \mathbb{R})$), one can regard W as a subgroup of $GL(n, \mathbb{Z})$. In this problem, you will prove that this is equivalent to our definition. One direction (part (a)) is easy; the other is challenging.

   (a) Prove that if W admits a crystallographic root system, then it can be realized as a subgroup of $GL(n, \mathbb{Z})$.

   (b) Let ζ be a primitive nth root of unity. (That is, ζ is a complex number such that $\zeta^n = 1$, but $\zeta^k \neq 1$ if $1 \leq k < n$.) For what values of n is ζ a solution to a quadratic equation with integer coefficients? (For example, the primitive 6th roots of unity are $e^{\pm 2\pi i/6} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. These both happen to satisfy the equation $x^2 - x - 1 = 0$.) (*Hint:* This question is related to part (c) of the previous question.)

   (c) Let $A, B \in GL(n, \mathbb{Z})$ be matrices corresponding to reflections of $\mathbb{R}^n$. Show that the group generated by A and B is a crystallographic dihedral group. (*Hint:* Study the eigenvalues of AB. You will need to use the previous part to complete your proof.)

(d) Prove that if W can be realized as a subgroup of $GL(n, \mathbb{Z})$, then it admits a crystallographic root system. (*Hint:* The question is trivial if W has rank 1, and you effectively did the rank 2 case in the previous part. Beyond that, you may wish to consider parabolic subgroups of W and proceed by induction on its rank.)

4. The root systems we have encountered so far should actually be called *reduced* root systems. The term "reduced" refers to the axiom that $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$. If we omit this condition, we get "possible nonreduced" root systems.

   Classify the nonreduced crystallographic root systems. (*Hint:* Even without our imposing a restriction on $\Phi \cap \mathbb{R}\alpha$, it turns out that as a consequence of the crystallographic condition, this set cannot be too big. First determine what possibilities there are for $\Phi \cap \mathbb{R}\alpha$—this will help you to construct some examples of nonreduced root systems. To prove that you have constructed all of them, it may be useful to first prove as a lemma that A_n does not admit a nonreduced root system.)