Problem Set 2a Hints

Due: September 27, 2005

1. Given a parabolic subgroup W' (generated by, say, $I \subset S$), let $V' = (\operatorname{span} \Delta')^\perp$, where $\Delta' \subset \Delta$ is the set of simple roots corresponding to I. It's clear that W' stabilizes V'; it remains to be shown that nothing outside of W' does, so that W' is equal to the stabilizer of V'. First show that it's enough to consider $w \in W \smallsetminus W'$ possessing a reduced expression ending with a generator not in I. If w ends in t, then by choosing $v \in V'$ such that $\langle v, \alpha_s \rangle > 0$ for all $s \notin I$, show that $\langle w^{-1}v, \alpha_t \rangle < 0$, so $w^{-1}v \neq v$, and w does not stabilize V'.

   Next, given a subspace $V' \subset V$ with stabilizer W', let Φ_1 be the set of roots orthogonal to V'. This is itself a root system. Choose positive and simple systems Π_1, Δ_1 for it, and let W_1 be the subgroup of W generated by the reflections in Φ_1 (or, equivalently, Δ_1). There are two things to prove: (1) W_1 is a parabolic subgroup, or, equivalently, that Δ_1 can be extended to a simple system Δ for the whole group; and (2) W_1 coincides with the stabilizer W'. Once we prove the first part, the second part is easy—we did the same thing in the previous paragraph. To prove (1), we have to choose a positive system Π for W that contains Π_1 (that's easy) with the additional property that no root in Δ_1 can be written as a positive linear combination of roots in $\Pi \smallsetminus \Pi_1$ (that guarantees that the roots in Δ_1 stay simple for W). Here's a sketch of one way to do that: let $V_1 = \operatorname{span} \Phi_1$, and extend V_1 to a hyperplane H_1 that contains no roots other than those in Φ_1. Take Π to be $\Pi_1 \cup (\text{the roots on one side of } H_1)$.

2. Crystallographic root systems. Let $N_{\alpha,\beta} = 2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$.

   (a) No hint.

   (b) In A_n and D_n, show that all reflections are conjugate to each other. This implies that all roots (in a given root system) must have the same length.

   (c) $I_2(m)$ contains a rotation by $2\pi/m$. Let α be a root, and let β be its image under this rotation. Then
   $$\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\|\alpha\|\|\beta\| \cos 2\pi/m}{\|\alpha\|^2} = 2\cos 2\pi/m.$$
   The requirement that $2\cos 2\pi/m \in \mathbb{Z}$ imposes severe restrictions on m—there are just four possible values for it. Verify that each one actually has a crystallographic root system. (*Note*: Three of the these are ones you've seen before—they're isomorphic to $A_1 \oplus A_1$, A_2, and B_2 (or C_2) respectively. The remaining crystallographic dihedral root system has its own name: G_2.)

   (d) First, note that in order for a reflection group to be crystallographic, every rank-2 parabolic subgroup of it must be a crystallographic dihedral group. Now, for each group satisfying this criterion, check whether $N_{\alpha,\beta} \in \mathbb{Z}$ when α and β are both simple roots in the geometric representation. If not, try to fix it up by changing the lengths of some roots. (This only happens for F_4.) Now, assuming that $N_{\alpha,\beta} \in \mathbb{Z}$ for all simple roots, you can show by induction on the length of w that $N_{w\alpha,\beta}$ and $N_{\alpha,w\beta}$ are integers, and then deduce that $N_{\alpha,\beta} \in \mathbb{Z}$ for all roots. (*Note*: It's also possible to try to give explicit descriptions of the crystallographic root systems for each of these groups. Humphreys works this out in detail.)

3. Reflection subgroups of $GL(n, \mathbb{Z})$.

   (a) Write W in the basis of simple roots.

   (b) If $e^{2\pi ij/n} = \cos 2\pi j/n + i \sin 2\pi j/n$ is a root of a quadratic polynomial with integer coefficients, then the other root must be its complex conjugate $e^{-2\pi ij/n}$, and the polynomial itself is
   $$(x - e^{2\pi ij/n})(x - e^{-2\pi ij/n}) = x^2 - (2\cos 2\pi j/n)x + 1.$$

So $2\cos 2\pi j/n \in \mathbb{Z}$.

(c) **Correction:** Assume that A and B generate a finite group. Now, A and B each fix a hyperplane in $\mathbb{R}^n$, so AB fixes the intersection of these two hyperplanes, which has dimension at least $n-2$. Thus, AB has $n-2$ eigenvalues equal to 1, so its characteristic polynomial factors as

$$(\lambda - 1)^{n-2} \cdot \text{(a quadratic)}.$$

The remaining eigenvalues must be roots of unity, since AB has finite order.

4. **Correction:** A_n doesn't admit a nonreduced root system if $n \geq 2$, but A_1 does. The question is basically asking: given a crystallographic root system, where can you add in additional multiples of existing roots without violating the crystallographic condition? The lemma implies that if a root belongs to a sub-root-system for a parabolic subgroup of type A_n ($n \geq 2$), then no additional multiples of that root are permitted. This leaves very few possibilities.