# Problem Set 3a

*Due: October 18, 2005*

In the study of groups and representations, it is often useful to have a method for taking a representation of a subgroup and producing from it a representation of the larger group. Here we will develop such a method for reflection groups, known as "truncated induction" or "MacDonald-Lusztig-Spaltenstein induction."

Let $W$ be a reflection group acting on $V$, and let $W'$ be a subgroup of $W$ that is also generated by reflections. Assume that $V$ is equipped with a $W-$ (and hence $W'$-) invariant inner product $\langle , \rangle$. Let $V'' = \{v \in V \mid wv = v \text{ for all } w \in W'\}$. amd let $V' = (V'')^{\perp}$. So $W'$ acts as an essential reflection group on $V'$. (*Note*: $W'$ is not necessarily a parabolic subgroup—it may or may not be equal to the full stabilizer of $V''$.) Let $S = \mathrm{Sym}(V^*)$, $S' = \mathrm{Sym}((V')^*)$. and $S'' = \mathrm{Sym}((V'')^*)$. $S^k$, $(S')^k$, and $(S'')^k$ denote the subspaces of homogeneous degree-$k$ polynomials in each of the preceding.

1. Abstractly, having a subspace $V' \subset V$ means that on the dual side, $(V')^*$ is naturally a quotient of $V^*$, and similarly $S'$ is a quotient of $S$. (The map $S \to S'$ is given by taking a polynomial function on $V$ and restricting its domain to $V'$.) However, since we have the decomposition $V = V' \oplus V''$, we can instead think of $S'$ as a subspace of $S$ by making the identification

   $$S' \simeq \{\text{polynomials in } S \text{ whose restriction to any plane of the form } x + V'' \text{ is constant}\}.$$

   Verify that $S'$ is isomorphic to this subspace of $S$. (You do not need to write anything up for this question; just think it though and make sure you understand it.)

Now, let $E'$ be an irreducible representation of $W'$ contained in $(S')^k$. (That is, $E' \subset (S')^k$ is stable under $W'$, but no proper subspace of $E'$ is stable.) Assume that there is no other subspace of $(S')^k$, and no subspace at all of $(S')^i$ for $0 \le i < k$, that is equivalent to $E'$ as a $W'$-representation. Now, let $E \subset S^k$ be the space spanned by all $W$-orbits of points in $E'$. (This makes sense since we have regarded $S'$ as a subspace of $S$.) In other words, $E$ is the smallest $W$-stable subspace of $S^k$ containing $E'$.

2. Show that there is a natural isomorphism

   $$S^n \simeq \bigoplus_{i+j=n} (S')^i \otimes (S'')^j$$

   that respects the action of $W'$. (Of course, $W'$ acts trivially on $S''$.) Our earlier identification of $S'$ with a subspace of $S$ corresponds to the piece $(S')^n \otimes (S'')^0$ in this decomposition.

3. Show that there is no subspace of $S^k$ other than $E'$ itself that is equivalent to $E'$ as a $W'$-representation.

4. Prove that $E$ is irreducible. (*Hint*: Show that if it were not, then $S^k$ would have to contain more than one copy of $E'$.)

5. Prove that there is no subspace of $S^i$ for $0 \le i < k$ that is equivalent to $E$ as a $W$-representation.

6. (Optional) The *sign representation* of a reflection group is the 1-dimensional representation $\mathrm{sgn} : W \to GL(1)$ given by $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. Show that the sign representation occurs in $S^N$ but not in $S^i$ for $0 \le i < N$ (here $N$ is the number of reflections in $W$).

It turns out that all irreducible representations of the symmetric group can be obtained by truncated induction of the sign representation from various parabolic subgroups. This is a fairly modern way to approach the representation theory of the symmetric group (truncated induction was invented in the 1970's, although the irreducible representations of the symmetric group were determined by other methods perhaps a century ago), and it's one that works well for other reflection groups as well (see Chapter 5 of Geck-Pfeiffer).