Problem Set 5

Due: November 22, 2005

Let W be the cyclic group of order n, acting as a complex reflection group on $\mathbb{C}$. Let s be a generator of W, acting on $\mathbb{C}$ by a primitive nth root of unity ζ.

Let $q_1, \ldots, q_n$ be a set of indeterminates, and let $A = \mathbb{Z}[\zeta][q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$. Finally, let

$$\mathcal{H} = A[T_s]/((T_s - q_1) \cdots (T_s - q_n))$$

1. Consider the case $n = 2$, where W is actually a finite Coxeter group. Then, the definition above doesn't quite match up with the Coxeter-group definition of the Hecke algebra. What is the relationship between the two? (Are they isomorphic? If not, can they be made isomorphic by some sort of change of scalars?)
2. If $\theta : A \to \mathbb{C}$ is the map defined by $\theta(q_i) = \zeta^i$, show that $\mathcal{H} \otimes_\theta \mathbb{C} \simeq \mathbb{C}[W]$.

Note: For Hecke algebras of Coxeter groups, we decided at some point to assume that we were working with algebras that possessed a specialization in which $a_s \mapsto 0$ and $b_s \mapsto 1$. We then remarked that after a suitable extension of scalars and change of variables, we could assume that $a_s = b_s - 1$; we also then changed the name of b_s to q_s. The analogue of this for the Hecke algebra of the cyclic group is to specialize $q_i \mapsto \zeta^i$ for $i = 1, \ldots, n-1$, but keep q_n as an indeterminate (but rename it q). The definition of $\mathcal{H}$ then becomes

$$\mathcal{H} = A[T_s]/((T_s - \zeta) \cdots (T_s - \zeta^{n-1})(T_s - q).$$

When $n = 2$, this definition *does* coincide with the Hecke algebra of a Coxeter group.

3. Show that $\mathcal{H}$ is a free A-module, with a basis $\{T_1, T_s, T_{s^2}, \ldots, T_{s^{n-1}}\}$, and the following multiplication rule:

$$T_s T_{s^k} = \begin{cases} T_{s^{k+1}} & \text{if } k+1 < n, \\ \sum \sigma_{n-i} T_{s^i} & \text{if } k+1 = n, \end{cases} \qquad \text{where } \sigma_k = (-1)^k \sum (\text{all } k\text{-fold products of indeterminates})$$

The rest of the problem set is quite hard. I've stated the problems for general n, but you should just try to do them in the special case $n = 3$ (or $n = 4$ if you want).

4. Show that $\mathcal{H}$ admits a symmetrizing trace that specializes to the "standard" symmetrizing trace of $\mathbb{C}[W]$, and compute the dual basis to the $\{T_i\}$ basis.
5. W has exactly n irreducible representations (up to isomorphism), all 1-dimensional. They are given by the formulas
$$\rho_i : W \to GL(1, \mathbb{C}), \qquad \rho_i(s) = [\zeta^i],$$
for $i = 0, 1, \ldots, n-1$. (Of course, ρ_0 is the trivial representation, and ρ_1 is the "reflection representation.") Find 1-dimensional modules for $\mathcal{H}$ that specialize to each of these.
6. Compute the Schur elements of all of these modules.

Historical Note: Although complex reflection groups were first classified in 1954, the idea of studying Hecke-like algebras associated to complex reflection groups really only came into being in the 1990's. Of course, before trying things out, it's impossible to know whether the concept even makes sense—maybe the algebras you try to define won't turn out to be free over A, or maybe they don't admit symmetrizing traces. At a conference on the Greek island of Spetses in 1993, M. Broué, G. Malle, and a few others decided to try things out in the easiest case: they worked out the Schur elements for the Hecke algebra of the group of order 3. Now you've done that as well.