Hour Exam 1 Solutions

February 15, 2005

1. (3 points each) Give examples (not taken from later in the exam) of the following types of differential equations: There are many possible answers to this question, of course. Here are some possibilities:

   (a) Separable:

   Solution: $5y'y^2 = \sin x$

   (b) First-order linear homogeneous:

   Solution: $y' + x^5 y = 0$

   (c) Third-order linear nonhomogeneous:

   Solution: $y''' + 5y'' - e^x y' + 2y = \cos x$

   (d) Second-order nonlinear:

   Solution: $y'' + y^2 = \tan x$

2. (4 points) What is the *complementary function* for a nonhomogeneous linear equation?

   Solution: the general solution to the associated homogeneous equation

3. (8 points) Find the general solution:
   $\dfrac{dy}{dx} = \dfrac{y^3}{x \ln x}$

   Solution:

$$\int \frac{dy}{y^3} = \int \frac{dx}{x \ln x}$$

Substitute $u = \ln x$, $du = \dfrac{dx}{x}$:
$$\frac{y^{-2}}{-2} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$$
$$y^{-2} = -2\ln|\ln x| + C$$
$$y = \frac{1}{\sqrt{-2\ln|\ln x| + C}}$$

4. (8 points) Find the integrating factor. Do not solve the equation.
   $xy' + 2y = 4e^x$

   Solution:

   Rewrite: $y' + \dfrac{2}{x}y = 4x^{-1}e^x$

   so
$$I(x) = e^{\int p(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = (e^{\ln x})^2 = x^2.$$

5. (8 points) Find the general solution.
   $y'' - y' - 12y = 0$

   Solution:

Auxiliary polynomial: $r^2 - r + 12 = 0$

$$(r-4)(r+3) = 0$$

$$r = 4, \quad -3$$

Particular solutions: $e^{4x}, \quad e^{-3x}$

General solution: $y = c_1 e^{4x} + c_2 e^{-3x}$

6. (8 points) Find the general solution.
   $y'' + 8y' + 25y = 0$

   Solution:

   Auxiliary polynomial: $r^2 + 8r + 25 = 0$

   $$r = \frac{-8 \pm \sqrt{64 - 100}}{2} = \frac{-8 \pm \sqrt{-36}}{2} = \frac{-8 \pm 6i}{2}$$

   $$r = -4 \pm 3i$$

   Particular solutions: $e^{-4x} \sin 3x, \quad e^{-4x} \cos 3x$

   General solution: $y = c_1 e^{-4x} \sin 3x + c_2 e^{-4x} \cos 3x$

7. (12 points) Solve the initial value problem.
   $y' + 2xy = 2x, \quad y(0) = 5$

   Solution:

   Integrating factor: $I(x) = e^{\int p(x)\,dx} = e^{\int 2x\,dx} = e^{x^2}$

   $$e^{x^2} y' + 2xe^{x^2} y = 2xe^{x^2}$$

   $$\frac{d}{dx}(e^{x^2} y) = 2xe^{x^2}$$

   $$e^{x^2} y = \int 2xe^{x^2}\,dx$$

   Substitute $u = x^2$, $du = 2x\,dx$ $\quad = \int e^u\,du = e^u + C = e^{x^2} + C$

   $$y = 1 + Ce^{-x^2}$$

   Plug in $y(0) = 5$: $5 = 1 + Ce^0 = 1 + C$

   $$C = 4$$

   Solution: $y = 1 + 4e^{-x^2}$

   The above solution is the one I expected most people to find, but it happens that this equation is also separable, so it can also be solved as follows:

$$y' = 2x - 2xy = 2x(1-y)$$

$$\frac{1}{1-y}\frac{dy}{dx} = 2x$$

$$\int \frac{dy}{1-y} = \int 2x\,dx$$

Substitute $u = 1-y$, $du = -dy$:

$$\int \frac{-du}{u} = \int 2x\,dx$$

$$-\ln|u| = -\ln|1-y| = x^2 + C_1$$

$$\ln|1-y| = -x^2 - C_1$$

$$|1-y| = e^{-x^2-C_1} = e^{-x^2}e^{-C_1}$$

Replace e^{-C_1} by C:

$$1-y = Ce^{-x^2}$$

$$y = 1 - Ce^{-x^2}$$

Plug in $y(0) = 5$:

$$5 = 1 - Ce^0 = 1 - C$$

$$C = -4$$

Solution:

$$y = 1 + 4e^{-x^2}$$

8. (8 points) The complementary function for the equation

$$y'' + 4y = 16x\cos 2x$$

is $y_c = c_1 \sin 2x + c_2 \cos 2x$. Write down the trial solution for this equation, but do not solve for its coefficients.

Solution: $y_p = A_0x^2\cos 2x + A_1x^2\sin 2x + A_2x\cos 2x + A_3x\sin 2x$

9. (12 points) Find y_p for the following equation. Do not find y_c. (The correct trial solution does not have any terms that also appear in y_c.)
$y'' - 4y = 15\cos x + 10e^{3x}$

Solution:

Trial solution:

$$y_p = A_0\cos x + A_1\sin x + A_2e^{3x}$$

$$y_p' = -A_0\sin x + A_1\cos x + 3A_2e^{3x}$$

$$y_p'' = -A_0\cos x - A_1\sin x + 9A_2e^{3x}$$

Plug in:

$$\begin{aligned} 15\cos x + 10e^{3x} &= y_p'' - 4y_p \\ 15\cos x + 10e^{3x} &= (-A_0\cos x - A_1\sin x + 9A_2e^{3x}) \\ &\qquad - 4(A_0\cos x + A_1\sin x + A_2e^{3x}) \\ &= -5A_0\cos x - 5A_1\sin x + 5A_2e^{3x} \end{aligned}$$

Match coefficients:

$$15 = -5A_0 \quad \text{so } A_0 = -3$$

$$0 = -5A_1 \quad \text{so } A_1 = 0$$

$$10 = 5A_2 \quad \text{so } A_2 = 2$$

Solution:

$$y_p = -3\cos x + 2e^{3x}$$

10. (20 points) Solve the initial value problem.
$y'' + 4y' + 4y = 6e^{-2x}, \quad y(0) = 2, \quad y'(0) = 1.$

Solution:

Associated homogeneous DE: $y'' + 4y' + 4y = 0$

Auxiliary polynomial:
$$\begin{aligned} r^2 + 4r + 4 &= 0 \\ (r+2)^2 &= 0 \\ r &= -2 \text{ (repeated root)} \end{aligned}$$

Complementary function: $y_c = c_1 e^{-2x} + c_2 x e^{-2x}$

Trial solution: $y_p = A_0 x^2 e^{-2x}$

(multiply by x^2 since both e^{-2x} and xe^{-2x} are in y_c)

$$\begin{aligned} y_p' &= -2A_0x^2e^{-2x} + 2A_0xe^{-2x} \\ y_p'' &= 4A_0x^2e^{-2x} - 4A_0xe^{-2x} - 4A_0xe^{-2x} + 2A_0e^{-2x} \\ &= 4A_0x^2e^{-2x} - 8A_0xe^{-2x} + 2A_0e^{-2x} \end{aligned}$$

Plug in:
$$\begin{aligned} 6e^{-2x} &= y_p'' + 4y + p' + 4y \\ &= 4A_0x^2e^{-2x} - 8A_0xe^{-2x} + 2A_0e^{-2x} \\ &\quad + 4(-2A_0x^2e^{-2x} + 2A_0xe^{-2x}) + 4(A_0x^2e^{-2x}) \\ &= 0A_0x^2e^{-2x} + 0A_0xe^{-2x} + 2A_0e^{-2x} \end{aligned}$$

so
$$\begin{aligned} 6 &= 2A_0 \\ A_0 &= 3 \end{aligned}$$

General solution:
$$\begin{aligned} y &= y_c + y_p = c_1e^{-2x} + c_2xe^{-2x} + 3x^2e^{-2x} \\ y' &= -2c_1e^{-2x} - 2c_2xe^{-2x} + c_2e^{-2x} - 6x^2e^{-2x} + 6xe^{-2x} \end{aligned}$$

Plug in $y(0) = 2$: $2 = c_1 \cdot 1 + c_2 \cdot 0 + 0 \implies c_1 = 2$

Plug in $y'(0) = 1$: $1 = -2(2)1 - 2c_2 \cdot 0 + c_2 \cdot 1 - 0 + 0 \implies c_2 = 5$

Final solution: $y = 2e^{-2x} + 5xe^{-2x} + 3x^2e^{-2x}$