Solutions to the Exercises in Section 5.2

1. The set of rational numbers (that is, numbers that can be written as fractions) is closed under addition: adding two fractions always gives you a fraction.

   It is not closed under scalar multiplication. In particular, if you multiply an irrational scalar (such as π or $\sqrt{2}$) by a rational number, the answer is irrational.

2. This question is either poorly worded or a trick question. Here are the two possibilities:

   They may have meant "the set of all $n \times n$ upper-triangular matrices" for some n. This set is closed under both addition and scalar multiplication.

   On the other hand, if they genuinely meant *all* upper-triangular matrices (regardless of size), then the addition operation isn't even always defined. For instance, you can't add a 2×2 upper-triangular matrix to a 5×5 one. Since addition isn't always defined, closure under addition doesn't make sense. However, this set is still closed under scalar multiplication.

3. Not closed under either addition or scalar multiplication, because the equation is nonhomogeneous.

4. Closed under both addition and scalar multiplication, because the equation is homogeneous.

5. Closed under both addition and scalar multiplication, because the equation is homogeneous.

6. (a) Yes. The zero vector in $M_2(\mathbb{R})$ is $\mathbf{0} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Since $\det \mathbf{0} = 0$, this matrix is an element of S.

   (b) Consider the matrices $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Clearly $\det A = \det B = 0$. However, their sum is given by $A + B = I_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, and $\det I_2 = 1$. So $A + B \notin S$.

   (c) Yes. Multiplying a 2×2 matrix by a scalar k changes its determinant by k^2. Now, if $A \in S$, then $\det A = 0$, so $\det(kA) = k^2 \det A = k^2 \cdot 0 = 0$. Since $\det(kA)$ is also 0, it follows that kA is an element of S.

7. (a) There is no zero vector.

   (b) Elements of $\mathbf{N}$ do not have additive inverses.

   (c) $\mathbf{N}$ is not closed under scalar multiplication. (For example, take the scalar $\frac{1}{2} \in \mathbb{R}$ and the element $1 \in \mathbf{N}$. Their product is not an element of $\mathbf{N}$.)

8. We discussed how to do this for $\mathbb{R}^n$ in class.

9. The zero vector is the matrix $\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$.

   The additive inverse of $\left[\begin{smallmatrix} a & b & c \\ d & e & f \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} -a & -b & -c \\ -d & -e & -f \end{smallmatrix}\right]$.

10. No. One reason is that it's not closed under addition. For example, take the elements $x^2 - 2x + 5$ and $-x^2 - 7x + 3$. Their sum is

$$(x^2 - 2x + 5) + (-x^2 - 7x + 3) = -9x + 8 \notin P.$$

11. Since this problem involves unusual operations, I will use the unusual symbols for operations that I've been using in class to avoid confusion. We need to decide whether $\mathbb{R}^2$ with the operations defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 y_2)$$
$$k \odot (x_1, x_2) = (kx_1, x_2^k)$$

is a vector space or not. It's actually quite close to being a vector space. It has a zero vector: it is $\mathbf{0} = (0, 1)$. The operation $\oplus$ is commutative and associative; also, associativity for $\odot$ holds, as do both kinds of distributivity. Most vectors have additive inverses: the additive inverse of (x_1, x_2) is $(-x_1, 1/x_2)$, since

$$(x_1, x_2) \oplus (-x_1, 1/x_2) = (0, 1) = \mathbf{0}.$$

However, this formula doesn't work for points in $\mathbb{R}^2$ whose second coordinate is 0. Those points don't have additive inverses, so $\mathbb{R}^2$ with these operations is not a vector space.

12. (1), (2), (5), (6), and (9) are true. The zero vector is the same as the usual one: it's $(0, 0)$. However, with this operation, every vector is *its own* additive inverse. Almost none of the "-tivity" axioms (commutativity, associativity, distributivity) hold, although (9) does.

13. There is a zero vector: it is $(1, 1)$. Most elements have additive inverses: the additive inverse of (x_1, x_2) is $(1/x_1, 1/x_2)$. However, points for which one or both coordinates are 0 do not have additive inverses.

14. (1), (2), (4), (5), (7), and (8) are true. This addition operation (matrix multiplication) is associative but not commutative. The zero vector is the identity matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Nonsingular matrices have additive inverses, but singular matrices don't. Neither distributive property holds.

15. (1), (2), (3), and (9) are true. There's no zero vector, because a zero vector would have to satisfy

$$\begin{aligned} A \oplus \mathbf{0} = -(A + \mathbf{0}) &= A \\ \mathbf{0} &= -2A \end{aligned}$$

(In the second line, "$-2A$" represents ordinary scalar multiplication, not $\bullet$.) But no matrix can be equal to $-2A$ for every matrix $A \in M_2(\mathbb{R})$.

16. This is similar to Problem 8.

17. This is similar to $M_n(\mathbb{R})$, which we discussed in class.

18. Yes. All the vector space axioms hold for $\mathbb{C}^3$ with complex scalars, and since the real numbers are a subset of the complex numbers, they also hold with real scalars. (When we learn about dimension in a couple of weeks, we'll see a difference in the two ways of thinking about $\mathbb{C}^3$: it's three-dimensional as a complex vector space, but six-dimensional as a real vector space.)