Math 1441 | Fall 2006

Sections 1 and 2 | P. Achar

Exam 1 Solutions

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Total points: 100 | Time limit: 80 minutes

No calculators, books, notes, or other aids are permitted. You must **show your work** and **justify your steps** to receive full credit.

1. (4 points each) Short answer:

   (a) What is the meaning of the derivative of a function? (Specifically, what information does it give you about the graph of the function?)

   Solution: It gives you the slopes of tangent lines to the graph of the function.

   (b) State the quotient rule.

   Solution: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

   (c) Let $h(x)$ be a function. Suppose you don't know a formula for $h(x)$, but you do know that $h'(x) = \sqrt{x^2 - 9}$. What is the slope of the tangent line to the graph of $h(x)$ at the point $(5, 3)$?

   Solution: The slope is $h'(5) = \sqrt{5^2 - 9} = \sqrt{25 - 9} = \sqrt{16} = 4$.

2. (6 points each) Evaluate the following limits:

   (a) $\lim_{x \to -1} \frac{x^2 - 36}{x - 4}$

   Solution:

   $$= \frac{(-1)^2 - 36}{(-1) - 4} = \frac{-35}{-5} = 7.$$

   (b) $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$

   Solution:

   $$= \lim_{x \to 2} \frac{(x-2)(x+3)}{(x-2)} = \lim_{x \to 2}(x+3) = 2 + 3 = 5.$$

   (c) $\lim_{t \to 5} \frac{\frac{1}{t} - \frac{1}{5}}{t - 5}$

   Solution:

   $$= \lim_{t \to 5} \frac{\frac{5}{5t} - \frac{t}{5t}}{t - 5} = \lim_{t \to 5} \frac{\frac{5-t}{5t}}{t - 5} \cdot \frac{5t}{5t} = \lim_{t \to 5} \frac{5 - t}{5t(t-5)} \cdot \frac{-1}{-1} = \lim_{t \to 5} \frac{t - 5}{-5t(t-5)}$$

   $$= \lim_{t \to 5} \frac{1}{-5t} = -\frac{1}{5 \cdot 5} = -\frac{1}{25}.$$

   (d) $\lim_{x \to \infty} \frac{8x^2 - 3x}{5x^3 - 2x + 3}$

   Solution:

   $$= \lim_{x \to \infty} \frac{8x^2 - 3x}{5x^3 - 2x + 3} \cdot \frac{1/x^3}{1/x^3} = \lim_{x \to \infty} \frac{8/x - 3/x^2}{5 - 2/x^2 + 3/x^3} = \frac{0 + 0}{5 + 0 + 0} = 0.$$

3. (8 points) Let $f(x) = 3x^2 - \dfrac{5}{x^2}$. Find $f''(x)$.

Solution:

Rewrite $f(x)$:	$f(x) = 3x^2 - 5x^{-2}$
Find the first derivative:	$f'(x) = 6x + 10x^{-3}$
Then find the second derivative:	$f''(x) = 6 - 30x^{-4}$

4. Consider the function $f(x) = x^2$.

(a) (6 points) Approximate the slope of the tangent line to the graph of $f(x)$ at the point $(-1, 1)$, using $\Delta x = 1$.

Solution: The formula for approximate slope is

$$\frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} \qquad \text{or} \qquad \frac{f(x+\Delta x) - f(x)}{\Delta x)}.$$

Using $x = -1$ and $\Delta x = 1$, this formula gives

$$\frac{f(-1+1) - f(-1)}{1} = \frac{f(0) - f(-1)}{1} = \frac{0^2 - (-1)^2}{1} = \frac{0-1}{1} = -1.$$

(b) (12 points) Find $f'(x)$ using the definition of derivative. You will receive no credit if you compute $f'(x)$ using the power rule.

Solution: The definition of the derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Thus, for $f(x) = x^2$, we have

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + (\Delta x)^2) - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{(2x+\Delta x)\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} (2x + \Delta x) \\ &= 2x. \end{aligned}$$

5. Consider the function $g(x) = x^2 + 6x\sqrt{x}$.

(a) (6 points) Find $g'(x)$.

Solution:

Rewrite $g(x)$:	$g(x) = x^2 + 6x \cdot x^{1/2} = x^2 + 6x^{3/2}$
so	$g'(x) = 2x + 6 \cdot \frac{3}{2}x^{1/2} = 2x + 9x^{1/2}$.

(b) (4 points) Find the slope of the tangent line to the graph of $g(x)$ at the point $(4, 64)$.

Solution:

$g'(4) = 2 \cdot 4 + 9 \cdot 4^{1/2} = 8 + 9 \cdot 2 = 26.$

(c) (10 points) Now, find an equation for that tangent line.

Solution: The line goes through the point $(4, 64)$ and has slope 26, so in point-slope form, its equation is

$$y - 64 = 26(x - 4).$$

Or, if you prefer to write it in slope-intercept form:

$$\begin{aligned} y - 64 &= 26x - 104 \\ y &= 26x - 40. \end{aligned}$$

6. (6 points each) Find the derivatives of the following functions.

(a) $f(x) = \dfrac{6x^2 - 2x + 2}{\sqrt{x}}$

Solution:

Rewrite $f(x)$: $\quad f(x) = (6x^2 - 2x + 2)x^{-1/2} = 6x^{3/2} - 2x^{1/2} + 2x^{-1/2}$

Then: $\quad \begin{aligned} f'(x) &= 6 \cdot \tfrac{3}{2}x^{1/2} - 2 \cdot \tfrac{1}{2}x^{-1/2} + 2 \cdot \left(-\tfrac{1}{2}\right) x^{-3/2} \\ &= 9x^{1/2} - x^{-1/2} - x^{-3/2}. \end{aligned}$

Or,

use the quotient rule: $\quad \begin{aligned} f'(x) &= \frac{\sqrt{x}\frac{d}{dx}(6x^2 - 2x + 2) - (6x^2 - 2x + 2)\frac{d}{dx}(\sqrt{x})}{(\sqrt{x})^2} \\ &= \frac{x^{1/2}(12x - 2) - (6x^2 - 2x + 2)(\frac{1}{2}x^{-1/2})}{x} \\ &= \frac{(12x^{3/2} - 2x^{1/2}) - (3x^{3/2} - x^{1/2} + x^{-1/2})}{x} \\ &= \frac{9x^{3/2} - x^{1/2} - x^{-1/2}}{x} \\ &= 9x^{1/2} - x^{-1/2} - x^{-3/2}. \end{aligned}$

(b) $h(x) = \dfrac{3x}{x - 4}$

Solution:

$$\begin{aligned} h'(x) &= \frac{(x - 4)\frac{d}{dx}(3x) - 3x\frac{d}{dx}(x - 4)}{(x - 4)^2} \\ &= \frac{(x - 4) \cdot 3 - 3x \cdot 1}{(x - 4)^2} = \frac{3x - 12 - 3x}{(x - 4)^2} \\ &= -\frac{12}{(x - 4)^2}. \end{aligned}$$

(c) $p(x) = (x^2 + 3x + 4)(x^3 - 4x)$ (Use the product rule; do not multiply out $p(x)$ beforehand.)

Solution:

$$\begin{aligned} p'(x) &= (x^2 + 3x + 4)\frac{d}{dx}(x^3 - 4x) + (x^3 - 4x)\frac{d}{dx}(x^2 + 3x + 4) \\ &= (x^2 + 3x + 4)(3x^2 - 4) + (x^3 - 4x)(2x + 3) \\ &= (3x^4 + 9x^3 + 12x^2 - 4x^2 - 12x - 16) + (2x^4 - 8x^2 + 3x^3 - 12x) \\ &= 5x^4 + 12x^3 - 24x - 16. \end{aligned}$$