Exam 2 Solutions
October 12, 2006

Total points: 100
Time limit: 80 minutes

No calculators, books, notes, or other aids are permitted. You must show your work and justify your steps to receive full credit.

1. Short answer:

(a) (4 points) What is the definition of critical point?

Solution: A point $x$ where $f'(x)$ is either 0 or undefined.

(b) (4 points) Suppose $f(x)$ is concave up on the interval $(-2, \infty)$, and that $x = 3$ is a critical point. Which of the choices below to describes this critical point? Briefly explain your answer.

   relative minimum relative maximum neither can’t tell

Solution: Relative minimum. The fact that $f(x)$ is concave up on $(-2, \infty)$ means that $f''(x)$ is positive on $(-2, \infty)$. In particular, $f''(3) > 0$, so the Second Derivative Test says that $f(x)$ has a relative minimum at $x = 3$.

(c) (4 points each) Sketch an example of a graph with the property that:

(i) it is increasing and concave down

Solution:

(ii) $x = 0$ is an inflection point but not a critical point

Solution:

2. (6 points each) Find the derivatives of the following functions. (It is not necessary to simplify your answers.)

(a) $f(x) = \left(\frac{1 + x}{1 - x}\right)^4$

Solution:

$f'(x) = 4 \left(\frac{1 + x}{1 - x}\right)^3 \frac{d}{dx} \left(\frac{1 + x}{1 - x}\right)$

chain rule

$= 4 \left(\frac{1 + x}{1 - x}\right)^3 \left(\frac{1 - (1 + x)}{(1 - x)^2}\right)$

quotient rule

$= 4 \left(\frac{1 + x}{1 - x}\right)^3 \left(\frac{2}{(1 - x)^2}\right)$

$= \frac{8(1 + x)^3}{(1 - x)^5}$

(b) $g(t) = (t + 1)^{3/2}t^4$

Solution:
\[ g(t) = (t + 1)^{3/2} \cdot 4t^3 + t^4 \cdot \frac{d}{dt}(t + 1)^{3/2} \quad \text{product rule} \]
\[ = (t + 1)^{3/2} \cdot 4t^3 + \frac{3}{2}(t + 1)^{1/2} \cdot 1 \quad \text{chain rule} \]
\[ = 4(t + 1)^{3/2}t^3 + \frac{3}{2}(t + 1)^{1/2}t^4 \]

3. (6 points each) In the following problems, find \( dy/dx \) by implicit differentiation.

(a) \( x^2 - y^2 = 8xy \)

Solution:
\[
2x - 2y \frac{dy}{dx} = 8x \frac{dy}{dx} + y \cdot 8 \quad \text{chain rule on } y^2; \text{ product rule on } 8xy
\]
\[-2y \frac{dy}{dx} - 8x \frac{dy}{dx} = 8y - 2x \]
\[-2y - 8x \frac{dy}{dx} = 8y - 2x \]
\[
\frac{dy}{dx} = \frac{8y - 2x}{2y - 8x} = \frac{2(4y - x)}{2(-y - 4x)}
\]
\[
\frac{dy}{dx} = \frac{4y - x}{-y - 4x}
\]

(b) \( (x + y)^{2/3} = x^2 \)

Solution:
\[
\frac{2}{3}(x + y)^{-1/3} \cdot \left( 1 + \frac{dy}{dx} \right) = 2x \quad \text{chain rule on left-hand side}
\]
\[
\frac{2}{3}(x + y)^{-1/3} + \frac{2}{3}(x + y)^{-1/3} \frac{dy}{dx} = 2x
\]
\[
\frac{2}{3}(x + y)^{-1/3} \frac{dy}{dx} = 2x - \frac{2}{3}(x + y)^{-1/3}
\]
\[
\frac{dy}{dx} = \frac{2x - \frac{2}{3}(x + y)^{-1/3}}{\frac{2}{3}(x + y)^{-1/3}} = \frac{2x - \frac{2}{3}(x + y)^{-1/3}}{\frac{2}{3}(x + y)^{1/3}} \cdot \frac{\frac{2}{3}(x + y)^{1/3}}{\frac{2}{3}(x + y)^{1/3}}
\]
\[
\frac{dy}{dx} = \frac{3x(x + y)^{1/3} - 1}{1} = 3x \sqrt[3]{x + y} - 1
\]

4. (10 points) Consider the function \( f(x) = x - 4 + 4/x \). Find the critical points of \( f(x) \). Then draw the sign chart for \( f'(x) \), and label each critical point on it with “min,” “max,” or “neither.”

Solution: First find the derivative: \( f(x) = x - 4 + 4x^{-1} \), so
\[
f'(x) = 1 - 4x^{-2} = 1 - 4/x^2.
\]
Critical points: \( f'(x) \) is undefined when \( x = 0 \), so \( x = 0 \) is a critical point. The other critical points are found by setting \( f'(x) = 0 \):
\[
1 - 4/x^2 = 0
\]
\[
1 = 4/x^2
\]
\[
x^2 = 4
\]
\[
x = \pm 2
\]
So the critical points are \( x = -2, 0, +2 \). Plug in some points:
\[
f'(-3) = 5/9 \quad f'(-1) = -3 \quad f'(1) = -3 \quad f'(3) = 5/9
\]
So the sign chart looks like this:

\[
\begin{array}{c|c|c|c|c}
 f'(x) & + & - & - & + \\
 -2 & 0 & 2 & \\
 \text{max} & \text{neither} & \text{min} & \\
\end{array}
\]

The First Derivative Test tells us that there is a relative maximum at \( x = -2 \), a relative minimum at \( x = 2 \), and no extremum at \( x = 0 \).

5. (10 points) Consider the function \( g(x) = x^5 - 5x^4 + 15x \).

(a) Find \( g'(x) \) and \( g''(x) \), and draw the sign chart for \( g''(x) \). Do not draw the sign chart for \( g'(x) \).

Solution: First and second derivatives:

\[
g'(x) = 5x^4 - 20x^3 + 15 \]
\[
g''(x) = 20x^3 - 60x^2
\]

There are no points where \( g''(x) \) is undefined. Setting \( g''(x) = 0 \), we find:

\[
20x^3 - 60x^2 = 0 \\
20x^2(x - 3) = 0 \\
x = 0, 3
\]

Plug in a few points: \( g''(-1) = -80 \), \( g''(1) = -40 \), \( g''(4) = 320 \). So the sign chart is:

\[
g''(x) \quad + \quad - \quad 0 \quad 3 \quad -
\]

(b) What are the inflection points of \( g(x) \)?

Solution: Only \( x = 3 \). (Note: \( x = 0 \) is not an inflection point because \( g''(x) \) does not change sign as it crosses \( x = 0 \).)

(c) On what intervals is \( g(x) \) concave up? Concave down?

Solution: Concave up on \((3, \infty)\). Concave down on \((-\infty, 0)\) and \((0, 3)\) (or on \((-\infty, 3)\)).

(d) The critical points of \( g(x) \) are at \( x = 1 \) and approximately \( x \approx 3.9514 \). Using the second derivative test, determine whether each of these is a relative minimum or a relative maximum. Warning: You will receive no credit if you answer this question using the first derivative test.

Solution: \( x = 1 \) is a relative maximum (because according to the sign chart, \( g''(1) < 0 \) and \( x \approx 3.9514 \) is a relative minimum (because \( g''(3.9514) > 0 \)).

6. (20 points) Find the maximum possible area of a rectangle whose perimeter is 36 cm.

(a) Sketch a picture, and label all the variables you will use.

Solution:

\[
x = \text{width of the rectangle} \\
y = \text{height of the rectangle} \\
A = \text{area of the rectangle}
\]

\[
\text{width} \quad | \quad \text{height} \\
\text{x} \quad | \quad y
\]

\[
\text{width} \quad | \quad \text{height} \\
\text{x} \quad | \quad y
\]
(b) Write down all the relationships between variables that you can find.

Solution:
\[ A = xy \quad \text{(area)} \]
\[ 36 = 2x + 2y \quad \text{(perimeter)} \]

(c) Which variable do you have to maximize? Write that variable as a function of just one other variable.

Solution: Need to maximize \( A \). Solve for \( y \) in the perimeter equation:
\[
2x + 2y = 36 \\
2y = 36 - 2x \\
y = \frac{36 - 2x}{2} = 18 - x
\]
and plug this in to the area equation:
\[
A = xy = x(18 - x) \\
A = 18x - x^2
\]

(d) Finish solving the problem. Clearly indicate your answer to the original question.

Solution: Derivative:
\[
\frac{dA}{dx} = 18 - 2x.
\]
Critical points: There are no points where \( \frac{dA}{dx} \) is undefined. Setting \( \frac{dA}{dx} = 0 \), we have:
\[
18 - 2x = 0 \\
-2x = -18 \\
x = 9
\]
So \( x = 9 \) is the only critical point. To make sure this is a relative maximum, draw the sign chart:
\[
\frac{dA}{dx} \quad + \quad 9 \quad -
\]
So there is indeed a maximum at \( x = 9 \).

The value of \( A \) when \( x = 9 \) is
\[
A(9) = 18(9) - 9^2 = 162 - 81 = 81,
\]
so the maximum possible area is 81 cm\(^2\).

7. (20 points) A person is blowing air into a spherical balloon at a rate of 5 cm\(^3\)/s. How fast is the radius of the balloon increasing when its radius is 5 cm?

(a) Sketch a picture, and label all the variables you will use.

Solution:
\[
\begin{align*}
& r = \text{radius of the balloon} \\
& V = \text{volume of the balloon} \\
& t = \text{time}
\end{align*}
\]
(b) Write down all the relationships between variables that you can find.

Solution:

\[ V = \frac{4}{3}\pi r^3 \quad \text{volume of a sphere} \]

\[ \frac{dV}{dt} = 5 \text{ cm}^3/\text{s} \quad \text{given in the problem} \]

(c) What derivative do you have to find to answer the question?

Solution: \( \frac{dr}{dt} \)

(d) Finish solving the problem. Clearly indicate your answer to the original question.

Solution: Implicit differentiation:

\[ V = \frac{4}{3}\pi r^3 \quad \text{volume of a sphere} \]

\[ \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \quad \text{chain rule on right-hand side} \]

\[ 5 = 4\pi (5)^2 \cdot \frac{dr}{dt} \quad \text{use } \frac{dV}{dt} = 5 \text{ and } r = 5 \]

\[ \frac{dr}{dt} = \frac{5}{4\pi \cdot 25} = \frac{1}{20\pi} \]

The radius is increasing at a rate of \( \frac{1}{20\pi} \text{ cm/s} \).