Math 1441 — Fall 2006

Sections 1 and 2 — P. Achar

Exam 2 Solutions

October 12, 2006

Total points: 100 — Time limit: 80 minutes

No calculators, books, notes, or other aids are permitted. You must **show your work** and **justify your steps** to receive full credit.

1. Short answer:

   (a) (4 points) What is the definition of *critical point*?

   Solution: A point x where $f'(x)$ is either 0 or undefined.

   (b) (4 points) Suppose $f(x)$ is concave up on the interval $(-2, \infty)$, and that $x = 3$ is a critical point. Which of the choices below to describes this critical point? Briefly explain your answer.

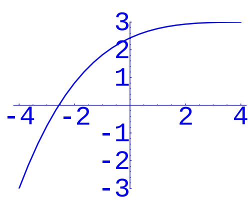
   relative minimum   relative maximum   neither   can't tell

   Solution: Relative minimum. The fact that $f(x)$ is concave up on $(-2, \infty)$ means that $f''(x)$ is positive on $(-2, \infty)$. In particular, $f''(3) > 0$, so the Second Derivative Test says that $f(x)$ has a relative minimum at $x = 3$.

   (c) (4 points each) Sketch an example of a graph with the property that:

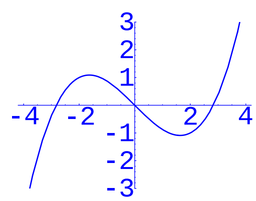
   (i) it is increasing and concave down

   Solution:



   (ii) $x = 0$ is an inflection point but not a critical point

   Solution:



2. (6 points each) Find the derivatives of the following functions. (It is not necessary to simplify your answers.)

   (a) $f(x) = \left(\dfrac{1+x}{1-x}\right)^4$

   Solution:

$$\begin{aligned}
f'(x) &= 4\left(\frac{1+x}{1-x}\right)^3 \frac{d}{dx}\left(\frac{1+x}{1-x}\right) && \text{chain rule}\\
&= 4\left(\frac{1+x}{1-x}\right)^3 \left(\frac{(1-x)\cdot 1 - (1+x)\cdot(-1)}{(1-x)^2}\right) && \text{quotient rule}\\
&= 4\left(\frac{1+x}{1-x}\right)^3 \left(\frac{2}{(1-x)^2}\right)\\
&= \frac{8(1+x)^3}{(1-x)^5}
\end{aligned}$$

   (b) $g(t) = (t+1)^{3/2} t^4$

   Solution:

$$\begin{aligned}
g'(t) &= (t+1)^{3/2} \cdot 4t^3 + t^4 \cdot \frac{d}{dt}(t+1)^{3/2} && \text{product rule} \\
&= (t+1)^{3/2} \cdot 4t^3 + t^4 \cdot \frac{3}{2}(t+1)^{1/2} \cdot 1 && \text{chain rule} \\
&= 4(t+1)^{3/2}t^3 + \frac{3}{2}(t+1)^{1/2}t^4
\end{aligned}$$

3. (6 points each) In the following problems, find dy/dx by implicit differentiation.

   (a) $x^2 - y^2 = 8xy$

   Solution:

$$\begin{aligned}
2x - 2y\frac{dy}{dx} &= 8x\frac{dy}{dx} + y \cdot 8 && \text{chain rule on } y^2\text{; product rule on } 8xy \\
-2y\frac{dy}{dx} - 8x\frac{dy}{dx} &= 8y - 2x \\
(-2y - 8x)\frac{dy}{dx} &= 8y - 2x \\
\frac{dy}{dx} &= \frac{8y - 2x}{-2y - 8x} = \frac{2(4y - x)}{2(-y - 4x)} \\
\frac{dy}{dx} &= \frac{4y - x}{-y - 4x}
\end{aligned}$$

   (b) $(x+y)^{2/3} = x^2$

   Solution:

$$\begin{aligned}
\frac{2}{3}(x+y)^{-1/3} \cdot \left(1 + \frac{dy}{dx}\right) &= 2x && \text{chain rule on left-hand side} \\
\frac{2}{3}(x+y)^{-1/3} + \frac{2}{3}(x+y)^{-1/3}\frac{dy}{dx} &= 2x \\
\frac{2}{3}(x+y)^{-1/3}\frac{dy}{dx} &= 2x - \frac{2}{3}(x+y)^{-1/3} \\
\frac{dy}{dx} &= \frac{2x - \frac{2}{3}(x+y)^{-1/3}}{\frac{2}{3}(x+y)^{-1/3}} = \frac{2x - \frac{2}{3}(x+y)^{-1/3}}{\frac{2}{3}(x+y)^{-1/3}} \cdot \frac{\frac{3}{2}(x+y)^{1/3}}{\frac{3}{2}(x+y)^{1/3}} \\
\frac{dy}{dx} &= \frac{3x(x+y)^{1/3} - 1}{1} = 3x\sqrt[3]{x+y} - 1
\end{aligned}$$

4. (10 points) Consider the function $f(x) = x - 4 + 4/x$. Find the critical points of $f(x)$. Then draw the sign chart for $f'(x)$, and label each critical point on it with "min," "max," or "neither."

   Solution: First find the derivative: $f(x) = x - 4 + 4x^{-1}$, so

$$f'(x) = 1 - 4x^{-2} = 1 - 4/x^2.$$

   Critical points: $f'(x)$ is undefined when $x = 0$, so $x = 0$ is a critical point. The other critical points are found by setting $f'(x) = 0$:

$$\begin{aligned}
1 - 4/x^2 &= 0 \\
1 &= 4/x^2 \\
x^2 &= 4 \\
x &= \pm 2
\end{aligned}$$

   So the critical points are $x = -2, 0, +2$. Plug in some points:

$$f'(-3) = 5/9 \qquad f'(-1) = -3 \qquad f'(1) = -3 \qquad f'(3) = 5/9$$

So the sign chart looks like this:

$f'(x)$	$+$	-2	$-$	0	$-$	2	$+$
		max		neither		min	

The First Derivative Test tells us that there is a relative maximum at $x = -2$, a relative minimum at $x = 2$, and no extremum at $x = 0$.

5. (10 points) Consider the function $g(x) = x^5 - 5x^4 + 15x$.

   (a) Find $g'(x)$ and $g''(x)$, and draw the sign chart for $g''(x)$. **Do not** draw the sign chart for $g'(x)$.

   Solution: First and second derivatives:

$$g'(x) = 5x^4 - 20x^3 + 15$$
$$g''(x) = 20x^3 - 60x^2$$

   There are no points where $g''(x)$ is undefined. Setting $g''(x) = 0$, we find:

$$\begin{aligned} 20x^3 - 60x^2 &= 0 \\ 20x^2(x-3) &= 0 \\ x &= 0, 3 \end{aligned}$$

   Plug in a few points: $g''(-1) = -80$, $g''(1) = -40$, $g''(4) = 320$. So the sign chart is:

$g''(x)$	$-$	0	$-$	3	$+$

   (b) What are the inflection points of $g(x)$?

   Solution: Only $x = 3$. (Note: $x = 0$ is not an inflection point because $g''(x)$ does not change sign as it crosses $x = 0$.)

   (c) On what intervals is $g(x)$ concave up? Concave down?

   Solution: Concave up on $(3, \infty)$. Concave down on $(-\infty, 0)$ and $(0, 3)$ (or on $(-\infty, 3)$)

   (d) The critical points of $g(x)$ are at $x = 1$ and approximately $x \approx 3.9514$. Using the second derivative test, determine whether each of these is a relative minimum or a relative maximum. *Warning:* You will receive no credit if you answer this question using the first derivative test.

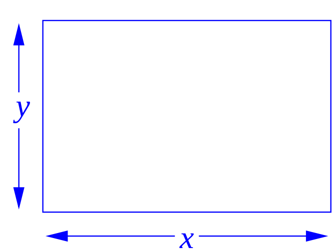
   Solution: $x = 1$ is a relative maximum (because according to the sign chart, $g''(1) < 0$) and $x \approx 3.9514$ is a relative minimum (because $g''(3.9514) > 0$).

6. (20 points) Find the maximum possible area of a rectangle whose perimeter is 36 cm.

   (a) Sketch a picture, and label all the variables you will use.

   Solution:

   x = width of the rectangle
   y = height of the rectangle
   A = area of the rectangle



(b) Write down all the relationships between variables that you can find.

Solution:

$$\begin{aligned} A &= xy \qquad &\text{(area)} \\ 36 &= 2x + 2y \qquad &\text{(perimeter)} \end{aligned}$$

(c) Which variable do you have to maximize? Write that variable as a function of just one other variable.

Solution: Need to maximize A. Solve for y in the perimeter equation:

$$\begin{aligned} 2x + 2y &= 36 \\ 2y &= 36 - 2x \\ y &= \frac{36 - 2x}{2} = 18 - x \end{aligned}$$

and plug this in to the area equation:

$$\begin{aligned} A &= xy = x(18 - x) \\ A &= 18x - x^2 \end{aligned}$$

(d) Finish solving the problem. Clearly indicate your answer to the original question.

Solution: Derivative:

$$\frac{dA}{dx} = 18 - 2x.$$

Critical points: There are no points where $\frac{dA}{dx}$ is undefined. Setting $\frac{dA}{dx} = 0$, we have:

$$\begin{aligned} 18 - 2x &= 0 \\ -2x &= -18 \\ x &= 9 \end{aligned}$$

So $x = 9$ is the only critical point. To make sure this is a relative maximum, draw the sign chart:

$$\frac{dA}{dx} \quad \begin{array}{c|c} + & - \\ \hline \multicolumn{2}{c}{9} \end{array}$$

So there is indeed a maximum at $x = 9$.
The value of A when $x = 9$ is

$$A(9) = 18(9) - 9^2 = 162 - 81 = 81,$$

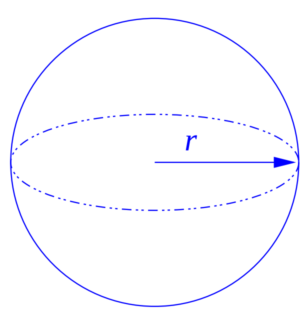
so the maximum possible area is 81 cm^2.

7. (20 points) A person is blowing air into a spherical balloon at a rate of 5 cm^3/s. How fast is the radius of the balloon increasing when its radius is 5 cm?

(a) Sketch a picture, and label all the variables you will use.

Solution:

r = radius of the balloon
V = volume of the balloon
t = time



(b) Write down all the relationships between variables that you can find.

Solution:

$V = \frac{4}{3}\pi r^3$ volume of a sphere

$\frac{dV}{dt} = 5 \text{ cm}^3/\text{s}$ given in the problem

(c) What derivative do you have to find to answer the question?

Solution: $\frac{dr}{dt}$

(d) Finish solving the problem. Clearly indicate your answer to the original question.

Solution: Implicit differentiation:

$$
\begin{aligned}
V &= \frac{4}{3}\pi r^3 && \text{volume of a sphere} \\
\frac{dV}{dt} &= 4\pi r^2 \cdot \frac{dr}{dt} && \text{chain rule on right-hand side} \\
5 &= 4\pi(5)^2 \cdot \frac{dr}{dt} && \text{use } \frac{dV}{dt} = 5 \text{ and } r = 5 \\
\frac{dr}{dt} &= \frac{5}{4\pi \cdot 25} = \frac{1}{20\pi}
\end{aligned}
$$

The radius is increasing at a rate of $\frac{1}{20\pi}$ cm/s.