Math 1441 | Fall 2006

Sections 1 and 2 | P. Achar

Exam 3 Solutions

November 9, 2006

Total points: 100 | Time limit: 80 minutes

No calculators, books, notes, or other aids are permitted. You must **show your work** and **justify your steps** to receive full credit.

1. (4 points each) Short answer:

   (a) Explain what is wrong with the following statement: "The antiderivative of the function $f(x) = 2x$ is $F(x) = x^2$."

   Solution: There is no such thing as *the* antiderivative; it should say: "*An* antiderivative ..."

   Alternatively, the statement could be corrected by saying, "The general antiderivative of $f(x) = 2x$ is $F(x) = x^2 + C$."

   (b) What method did we use in class to find the derivatives of inverse trigonometric functions (such as $\arcsin x$) after we already knew the derivatives of the ordinary trigonometric functions? (*Hint*: It's the same technique we used to find $\frac{d}{dx} a^x$ when we already knew $\frac{d}{dx} \log_a x$.)

   Solution: implicit differentiation.

2. (5 points) Find the derivative of $f(x) = \tan x$ using the derivatives of $\sin x$ and $\cos x$ and the quotient rule. You will receive no credit if you simply write down the derivative of $\tan x$ without showing your work.

   Solution:

$$
\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} \\
\frac{d}{dx}\tan x &= \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{(\cos x)^2} && \text{quotient rule} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} && \text{use the identity } \sin^2 x + \cos^2 x = 1 \\
&= \sec^2 x
\end{aligned}
$$

3. (4 points each) Find the derivatives of the following functions **without using the chain rule**. (That is, use algebra to first change the functions into a form in which the chain rule is not required.) You must clearly indicate how you are rewriting $f(x)$ in order to receive full credit.

   (a) $f(x) = \tan(\arctan e^x)$

   Solution: By the definition of arctan, we have $\tan(\arctan e^x) = e^x$. That is, $f(x) = e^x$, so $f'(x) = e^x$.

   (b) $f(x) = \ln x^3$

   Solution: By one of the properties of logarithms, $\ln x^3 = 3 \ln x$. So $f(x) = 3 \ln x$, and $f'(x) = 3 \cdot \frac{1}{x} = 3/x$.

4. (10 points) Find the equation of the tangent line to the curve $y = \sin(x^2)$ at the point $(\sqrt{\pi}, 0)$.

Solution: First, find the derivative:

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x \qquad \text{chain rule}$$

so at $x = \sqrt{\pi}$, the slope of the tangent line is:

$$m = \cos((\sqrt{\pi})^2) \cdot 2\sqrt{\pi} = \cos\pi \cdot 2\sqrt{\pi} = -1 \cdot 2\sqrt{\pi} = -2\sqrt{\pi}.$$

In point-slope form, the equation of the tangent line is

$$y - 0 = -2\sqrt{\pi}(x - \sqrt{\pi})$$

Rearranging a bit, we have

$$\begin{aligned} y &= -2\sqrt{\pi}x + 2\sqrt{\pi}\sqrt{\pi} \\ y &= -2\sqrt{\pi}x + 2\pi. \end{aligned}$$

5. (10 points) Let $F(x)$ be the antiderivative of $f(x) = 3x^2 + 5$ such that $F(1) = 10$. Find $F(x)$.

Solution: The general antiderivative of $f(x) = 3x^2 + 5$ is

$$\int (3x^2 + 5)\, dx = x^3 + 5x + C.$$

Since $F(x) = x^3 + 5x + C$, and $F(1) = 10$, we have:

$$\begin{aligned} 1^3 + 5(1) + C &= 10 \\ 6 + C &= 10 \\ C &= 4 \end{aligned}$$

Thus, $F(x) = x^3 + 5x + 4$.

6. (5 points) Find $\frac{d}{dx}(\sec x + \operatorname{arccot} x + \log_5 x + 6^x)$.

Solution:

$$\begin{aligned} &\frac{d}{dx}(\sec x + \operatorname{arccot} x + \log_5 x + 6^x) \\ &= \sec x \tan x + \frac{-1}{1+x^2} + \frac{1}{x \ln 5} + 6^x \ln 6 \end{aligned}$$

7. (6 points each) Find the derivatives of the following functions.

(a) $f(x) = \ln(\arccos^2 x)$

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{\arccos^2 x} \frac{d}{dx} \arccos^2 x && \text{chain rule} \\ &= \frac{1}{\arccos^2 x} \cdot 2 \arccos x \frac{d}{dx} \arccos x && \text{chain rule} \\ &= \frac{1}{\arccos^2 x} \cdot 2 \arccos x \cdot \frac{-1}{\sqrt{1-x^2}} \\ &= -\frac{2 \arccos x}{\sqrt{1-x^2} \arccos^2 x} = -\frac{2}{\sqrt{1-x^2} \arccos x} \end{aligned}$$

(b) $g(x) = (\cos x)e^{x^2}$

Solution:

$$\begin{aligned} g'(x) &= \cos x \frac{d}{dx}e^{x^2} + e^{x^2}\frac{d}{dx}\cos x && \text{product rule} \\ &= \cos x\, e^{x^2} \cdot 2x + e^{x^2}(-\sin x) && \text{chain rule on } e^{x^2} \\ &= 2xe^{x^2}\cos x - e^{x^2}\sin x \end{aligned}$$

(c) $h(x) = \dfrac{2\cot^2 x}{3e^{4x} - x}$

Solution:

$$\begin{aligned} h'(x) &= \frac{(3e^{4x} - x)\frac{d}{dx}(2\cot^2 x) - 2\cot^2 x\, \frac{d}{dx}(3e^{4x} - x)}{(3e^{4x} - x)^2} && \text{quotient rule} \\ &= \frac{(3e^{4x} - x) \cdot 4\cot x \cdot (-\csc^2 x) - 2\cot^2 x \cdot (3e^{4x} \cdot 4 - 1)}{(3e^{4x} - x)^2} && \text{chain rule on } 2\cot^2 x \text{ and } e^{4x} \\ &= \frac{-4(3e^{4x} - x)\cot x \csc^2 x - 2(12e^{4x} - 1)\cot^2 x}{(3e^{4x} - x)^2} \end{aligned}$$

8. (6 points each) Evaluate the following indefinite integrals.

(a) $\displaystyle\int (\sin x + \sec x \tan x)\, dx$

Solution: $= -\cos x + \sec x + C$

(b) $\displaystyle\int \left(x^3 + 3x - \frac{5}{x^4}\right) dx$

Solution:

$$\begin{aligned} &= \int (x^3 + 3x - 5x^{-4})\, dx \\ &= \frac{x^4}{4} + \frac{3x^2}{2} - 5\frac{x^{-3}}{-3} + C \\ &= \frac{x^4}{4} + \frac{3x^2}{2} + \frac{5}{3x^3} + C \end{aligned}$$

(c) $\displaystyle\int \left(5e^x + \frac{1}{x}\right) dx$

Solution: $= 5e^x + \ln|x| + C$

9. (6 points each) Evaluate the following indefinite integrals.

(a) $\displaystyle\int x\sqrt[3]{5x^2 - 1}\, dx$

Solution:

$$= \int x(5x^2 - 1)^{1/3}\, dx$$

Substitution:

$$\begin{aligned} u &= 5x^2 - 1 \\ du &= 10x\, dx \\ \tfrac{1}{10}\, du &= x\, dx \end{aligned}$$

$$\begin{aligned} &= \int u^{1/3} \cdot \frac{1}{10}\, du \\ &= \frac{1}{10}\frac{u^{4/3}}{4/3} + C = \frac{3}{40}u^{4/3} + C \\ &= \frac{3}{40}(5x^2 - 1)^{4/3} + C \end{aligned}$$

(b) $\displaystyle\int \frac{6x+1}{(3x^2+x-2)^2}\,dx$

Solution:

$$= \int \frac{1}{u^2}\,du \qquad \text{Substitution: } \begin{aligned} u &= 3x^2+x-2 \\ du &= (6x+1)\,dx \end{aligned}$$

$$= \int u^{-2}\,du$$

$$= \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C$$

$$= -\frac{1}{3x^2+x-2} + C$$

(c) $\displaystyle\int \csc 7x \cot 7x\,dx$

Solution:

$$= \int \csc u \cot u \cdot \frac{1}{7}\,du \qquad \text{Substitution: } \begin{aligned} u &= 7x \\ du &= 7\,dx \\ \tfrac{1}{7}\,du &= dx \end{aligned}$$

$$= -\frac{1}{7}\csc u + C$$

$$= -\frac{1}{7}\csc 7x + C$$