Basic Facts on Sheaves

Definition 1. A **sheaf of abelian groups** $\mathcal{F}$ on a topological space X is the following collection of data:

- for each open set $U \subset X$, an abelian group $\mathcal{F}(U)$, with $\mathcal{F}(\varnothing) = 0$
- if $U \subset V$, a **restriction map** $\rho_{VU} : \mathcal{F}(V) \to \mathcal{F}(U)$, with $\rho_{UU} = \mathrm{id}$

such that

(1) (restriction) if $U \subset V \subset W$, then $\rho_{WU} = \rho_{VU} \circ \rho_{WV}$
(2) (gluing) if $\{V_i\}$ is an open covering of U, and we have $s_i \in \mathcal{F}(V_i)$ such that for all i, j, $\rho_{V_i,V_i\cap V_j}(s_i) = \rho_{V_j,V_i\cap V_j}(s_j)$, then there exists a unique $s \in \mathcal{F}(U)$ such that $\rho_{U,V_i}(s) = s_i$ for all i.

If one omits the gluing condition from the above definition, one has a **presheaf of abelian groups**. Elements of $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over U, and elements of $\mathcal{F}(X)$ are called **global sections**.

One can also define (pre)sheaves of R-modules, vector spaces, *etc.*

Notation 2. $\Gamma(U, \mathcal{F}) := \mathcal{F}(U)$. $s|_V := \rho_{VU}(s)$.

Lemma 3. *Let $\mathcal{F}$ be a sheaf. If $\{V_i\}$ is an open cover of U, and $s, t \in \mathcal{F}(U)$ are sections such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$. In particular, if $s|_{V_i} = 0$ for all i, then $s = 0$.*

Definition 4. Let $\mathcal{F}$ be a presheaf on X. The **stalk** of $\mathcal{F}$ at a point $x \in X$, denoted $\mathcal{F}_x$, is the group whose elements are equivalence classes of pairs

$$(U, s) \qquad \text{where } U \text{ is a neighborhood of } x \text{ and } s \in \mathcal{F}(U),$$

and the equivalence relation is

$$(U, s) \sim (V, t) \qquad \text{if there is an open set } W \subset U \cap V \text{ with } x \in W \text{ and } s|_W = t|_W$$

For any neighborhood U of x, there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$ sending a section $s \in \mathcal{F}(U)$ to the equivalence class of the pair (U, s). That equivalence class is denoted s_x and is called the **germ** of s at x.

Lemma 5. *A section is determined by its germs. That is, if $s, t \in \mathcal{F}(U)$, and if $s_x = t_x$ for all $x \in U$, then $s = t$.*

Definition 6. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves. A **morphism** $f : \mathcal{F} \to \mathcal{G}$ is a collection of abelian group homomorphisms $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$, where U ranges over all open sets of X, that are compatible with restriction. That is, the following diagram must commute whenever $V \subset U$:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho^{\mathcal{F}}_{UV}"'] & \mathcal{G}(U) \arrow[d, "\rho^{\mathcal{G}}_{UV}"] \\ \mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V) \end{tikzcd}$$

A morphism $f : \mathcal{F} \to \mathcal{G}$ is an **isomorphism** if there is another morphism $g : \mathcal{G} \to \mathcal{F}$ such that $f \circ g = \mathrm{id}_{\mathcal{G}}$ and $g \circ f = \mathrm{id}_{\mathcal{F}}$.

It is easy to check that $f : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all open sets $U \subset X$.

Definition 7. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves, and let $x \in X$. The **induced homomorphism of stalks** at x is the map $f_x : \mathcal{F}_x \to \mathcal{G}_x$ given by $f_x([(U, s)]) = [(U, f_U(s))]$. (Here, the notation $[(V, t)]$ for an element of a stalk denotes the equivalence class of the pair (V, t).)

One must check that f_x is well-defined; the proof uses the fact that the f_U are compatible with restriction.

Definition 8. A **sub(pre)sheaf** of $\mathcal{F}$ is a (pre)sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open set U, and whose restriction maps are the restrictions of the restriction maps of $\mathcal{F}$. That is, in the diagram

$$\begin{tikzcd} \mathcal{G}(U) \arrow[r, hook] \arrow[d, "\rho^{\mathcal{G}}_{UV}"'] & \mathcal{F}(U) \arrow[d, "\rho^{\mathcal{F}}_{UV}"] \\ \mathcal{G}(V) \arrow[r, hook] & \mathcal{F}(V) \end{tikzcd}$$

we must have $\rho^{\mathcal{G}}_{UV} = \rho^{\mathcal{F}}_{UV}|_{\mathcal{G}(U)}$.

Definition 9. Let $\mathcal{F}$ be a presheaf. The **sheafification** of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \text{for all } x \in U,\ s(x) \in \mathcal{F}_x, \text{ and there is a neighborhood } V \subset U \\ \text{of } x \text{ and a section } t \in \mathcal{F}(V) \text{ such that for all } y \in V,\ s(y) = t_y \end{array} \right\}$$

(Check that $\mathcal{F}^+$ really is a sheaf.)

When $\mathcal{F}$ happens to be a subpresheaf of a sheaf, there is a convenient alternate description of sheafification.

Lemma 10. *Suppose $\mathcal{F}$ is a subpresheaf of $\mathcal{G}$, and assume that $\mathcal{G}$ is a sheaf. Define a new sheaf $\mathcal{F}' \subset \mathcal{G}$ by*

$$\mathcal{F}'(U) = \{ s \in \mathcal{G}(U) \mid \textit{there is an open cover } \{V_i\} \textit{ of } U \textit{ such that } s|_{V_i} \in \mathcal{F}(U) \textit{ for all } i \}.$$

Then $\mathcal{F}' \simeq \mathcal{F}^+$.

Corollary 11. *If $\mathcal{F}$ is a sheaf, then $\mathcal{F} \simeq \mathcal{F}^+$.*

Lemma 12. *Let $\mathcal{F}$ be a presheaf. For all $x \in X$, $\mathcal{F}_x \simeq \mathcal{F}^+_x$.*

For any presheaf $\mathcal{F}$, there is a canonical morphism $\iota : \mathcal{F} \to \mathcal{F}^+$ defined as follows: given $s \in \mathcal{F}(U)$, we let $\iota_U(s) \in \mathcal{F}^+(U)$ be the function $U \to \coprod \mathcal{F}_x$ given by $\iota_U(s)(x) = s_x$.

Lemma 13 (Universal property of sheafification)**.** *Let $\mathcal{F}$ be a presheaf and $\mathcal{G}$ a sheaf. Given a morphism $f : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $f^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $f = f^+ \circ \iota$.*

In other words, this lemma asserts the existence of a morphism f^+ making the following diagram commute:

$$\begin{tikzcd} & \mathcal{F}^+ \arrow[dr, dotted, "f^+"] & \\ \mathcal{F} \arrow[ur, "\iota"] \arrow[rr, "f"'] & & \mathcal{G} \end{tikzcd}$$

Definition 14. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The **kernel** of f is the sheaf $\ker f$ given by $(\ker f)(U) = \ker f_U$. This is a subsheaf of $\mathcal{F}$.

(The assertion that $\ker f$ is a sheaf requires proof.)

Definition 15. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The **presheaf-image** of f is the presheaf $\operatorname{psim} f$ given by $(\operatorname{psim} f)(U) = \operatorname{im} f_U$. This is a subpresheaf of $\mathcal{G}$, but it is not in general a sheaf.

The **image** of f is the sheafification of its presheaf-image: $\operatorname{im} f = (\operatorname{psim} f)^+$. By Lemma 10, $\operatorname{im} f$ can be identified with a subsheaf of $\mathcal{G}$.

Definition 16. A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is **injective** if $\ker f = 0$, and **surjective** if $\operatorname{im} f = \mathcal{G}$.

Lemma 17. *A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is injective if and only if $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set U.*

WARNING: The analogue of the preceding lemma for surjective morphisms is false.

Lemma 18. *A morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective, an isomorphism) if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective, an isomorphism) for all $x \in X$.*