# Goresky–MacPherson Stratifications and Perversities

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Intersection cohomology complexes are in some sense the most important perverse sheaves—essentially all perverse sheaves that come up in practice are direct sums of intersection cohomology complexes. But as pointed out in the last set of notes, it is not clear that intersection cohomology complexes are actually constructible. In fact, we need to modify the definition of "stratification" to make sure that they are.

**Definition 1.** Let $X$ be a topological space equipped with a stratification $\mathcal{S}$, and let $n = \max\{\dim S \mid S \in \mathcal{S}\}$. $\mathcal{S}$ is called a **topological stratification** (or a **Goresky–MacPherson stratification**) if it satisfies the following additional condition: for any point $x$ in a stratum of dimension $k$ with $k < n$, there is a neighborhood $U$ together with a homeomorphism

$$U \simeq \mathbb{R}^k \times C$$

where

$$C = (Y \times [0, \infty)/(Y \times \{0\})$$

(the "cone" on $Y$), and where $Y$ is a topologically stratified space of dimension $n - k - 1$. Moreover, the set of strata $\mathcal{T}$ of $Y$ should be in bijection with the strata of $X$ that are larger than $S$ in the partial order on strata: specifically, if $S' \in \mathcal{S}$ is such that $S \subset \overline{S'}$ but $S \neq S'$, then there must exist a stratum $T$ of $Y$ such that the homeomorphism $U \simeq \mathbb{R}^k \times C$ restricts to a homeomorphism

$$S' \cap U \simeq \mathbb{R}^k \times (T \times (0, \infty)),$$

and every $T \in \mathcal{T}$ must arise in this way from some $S'$.

**Theorem 2.** *If $\mathcal{S}$ is a topological stratification of $X$, then for any stratum $S$ and any local system $\mathcal{E}$ on $S$, the sheaf $Ri_{S*}\mathcal{E}$ (where $i_S : S \hookrightarrow X$ is inclusion) is constructible.*

Now we can go back and revisit the theorems on gluing of $t$-structures and on middle extension. Those theorems were stated in terms of categories $D$, $D_U$, and $D_Z$, together with six pull-back and push-forward functors. On a topologically stratified space, the relevant pull-back and push-forward functors all take constructible sheaves to constructible sheaves, so we can invoke those theorems with $D = D^b_c(X)$, $D_U = D^b_c(U)$, and $D^b_c(Z)$. In particular, we can define a "perverse $t$-structure" on $D^b_c(X)$ (not just on the larger category $D^b(X)$), and we can revise the definition of "perverse sheaves" as follows.

**Definition 3.** The category of **perverse sheaves** on $X$ (with respect to a topological stratification $\mathcal{S}$ and a perversity function $p : \mathcal{S} \to \mathbb{Z}$), denoted $M(X)$, is the heart of the **perverse *t*-structure** on $D^b_c(X)$ given by

$$
\begin{aligned}
{}^pD^b_c(X)^{\le 0} &= \{\mathcal{F} \mid i_S^{-1}\mathcal{F} \in {}^{\mathrm{std}}D^b_c(S)^{\le p(S)} \text{ for all } S \in \mathcal{S}\} \\
{}^pD^b_c(X)^{\ge 0} &= \{\mathcal{F} \mid i_S^{!}\mathcal{F} \in {}^{\mathrm{std}}D^b_c(S)^{\ge p(S)} \text{ for all } S \in \mathcal{S}\}
\end{aligned}
$$

Then, revisiting the middle extension theorem gives the following result.

**Corollary 4.** *On a topologically stratified space, the intersection cohomology complexes $\mathrm{IC}(\overline{S}, \mathcal{E})$ are constructible.*

Recall that a **simple** or **irreducible object** $A$ in an abelian category is an object with the property that any monomorphism $A' \to A$ is either 0 or an isomorphism. Equivalently, any epimorphism $A \to A''$ must be either 0 or an isomorphism.

We call a local system "simple" if it has no nontrival ordinary subsheaf that is also a local system. A simple local system on a stratum of a stratified space is a simple object in the category of ordinary constructible sheaves, but not in the category of all ordinary sheaves.

**Theorem 5.** *The simple objects in the category $M(X)$ are precisely the intersection cohomology complexes $\mathrm{IC}(\overline{S}, \mathcal{E})$ where a $\mathcal{E}$ is a simple local system on $S$.*

So far, perversity functions have been arbitrary integer-valued functions on the set of strata. In the literature, however, one usually imposes a few mild conditions on perversities, and these conditions have a number of useful consequences.

**Definition 6.** A perversity function $p : \mathcal{S} \to \mathbb{Z}$ is said to be **of Goresky–MacPherson type** if the following conditions hold:

(1) $p(S)$ depends only on the dimension of $S$: that is, there is a function $\tilde{p} : \mathbb{N} \to \mathbb{Z}$ such that $p(S) = \tilde{p}(\dim S)$.
(2) $\tilde{p}$ is weakly decreasing.
(3) $\tilde{p}^* : \mathbb{N} \to \mathbb{Z}$, defined by $\tilde{p}^*(k) = -k - \tilde{p}(k)$, is also weakly decreasing.

The last two conditions are equivalent to requiring that

$$0 \leq \tilde{p}(l) - \tilde{p}(k) \leq k - l$$

whenever $0 \leq l \leq k$.

Often, Goresky–MacPherson perversities are normalized by also requiring $\tilde{p}(0) = 0$. With this normalization, it is clear that the Goresky–MacPherson perversity with the smallest possibles values (the "bottom perversity") is

$$\tilde{p}(n) = -n.$$

Conversely, the "top perversity" is the constant perversity

$$\tilde{p}(n) = 0.$$

(As we have noted before, perverse sheaves with respect to a constant perversity function are the same as ordinary constructible sheaves.) In between are the "upper middle" and "lower middle" perversities

$$\tilde{p}(n) = \lceil -n/2 \rceil \qquad \text{and} \qquad \tilde{p}(n) = \lfloor -n/2 \rfloor.$$

On a space with only even-dimensional strata, these two perversities coincide with each other and with what we have previously called the "middle perversity."