Operations on Sheaves; Adjointness Theorems

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Definition 1. Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathcal{F}$ is a sheaf on X, the **push-forward** by f of $\mathcal{F}$, also called its **direct image**, and denoted $f_*\mathcal{F}$, is the sheaf on Y defined by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$.

Remark 2. Note that specifying a sheaf on a one-point topological space is the same as specifying a single abelian group. One-point spaces occur in several examples below, and we will frequently treat abelian groups as sheaves on these spaces without further comment.

Example 3. Let $f : X \to \{x\}$ be the constant map to a one-point space. Then $f_*\mathcal{F} \simeq \Gamma(X, \mathcal{F})$ for any sheaf $\mathcal{F}$ on X.

Example 4. Let $x_0 \in X$, and let $i : \{x_0\} \hookrightarrow X$ be the inclusion of the point. Let A be an abelian group, thought of as a sheaf on $\{x_0\}$. Then i_*A is sheaf on X whose stalks are all 0 except at x_0, where its stalk is isomorphic to A. This kind of sheaf is called a **skyscraper sheaf**.

Definition 5. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Their **direct sum** is the sheaf $\mathcal{F} \oplus \mathcal{G}$ defined by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ for all open sets $U \subset X$.

Example 6. Let $X = \mathbb{C} \smallsetminus \{0\}$, and let $f : X \to X$ be the map $f(z) = z^2$. Then $f_*\underline{\mathbb{C}} \simeq \underline{\mathbb{C}} \oplus \mathcal{Q}$, where $\mathcal{Q}$ is the square-root sheaf on X.

Definition 7. Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathcal{F}$ is a sheaf on Y, its **pull-back** or **inverse image**, denoted $f^{-1}\mathcal{F}$, is the sheafification of the presheaf $\mathrm{ps}f^{-1}\mathcal{F}$ defined by

$$\begin{aligned}(\mathrm{ps}f^{-1}\mathcal{F})(U) &= \varinjlim_{V \supset f(U)} \mathcal{F}(V) \\ &= \text{equivalence classes of pairs } (V, s) \text{ where } V \supset f(U) \text{ and } s \in \mathcal{F}(V).\end{aligned}$$

The equivalence relation is the same as for stalks: $(V, s) \sim (V', s')$ if there is an open set $W \subset V \cap V'$ such that $s|_W = s'|_W$.

Example 8. For any continuous map $f : X \to Y$ and any abelian group A, we have $f^{-1}\underline{A}_Y \simeq \underline{A}_X$.

Definition 9. Let $i : X \hookrightarrow Y$ be the inclusion map of a subspace. If $\mathcal{F}$ is a sheaf on Y, the pull-back $i^{-1}\mathcal{F}$ is also called the **restriction** of $\mathcal{F}$ to X, and is denoted $\mathcal{F}|_X$.

Notation 10. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. The set of all morphisms of sheaves $\mathcal{F} \to \mathcal{G}$ is denoted $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$, or simply $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$. It is an abelian group.

Definition 11. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and let $S : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{A}$ be functors. We say that S is **left-adjoint** to T and that T is **right-adjoint** to S, or simply that (S, T) is an **adjoint pair**, if

$$\mathrm{Hom}_{\mathcal{B}}(S(A), B) \simeq \mathrm{Hom}_{\mathcal{A}}(A, T(B))$$

for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Example 12. Perhaps the best-known adjoint pair of functors is the following. Let $\mathcal{A} = \mathcal{B} =$ the category of abelian groups. For a fixed abelian group C, the functors $\cdot \otimes C$ and $\mathrm{Hom}(C, \cdot)$ are adjoint:

$$\mathrm{Hom}(A \otimes C, B) \simeq \mathrm{Hom}(A, \mathrm{Hom}(C, B)).$$

Theorem 13. *Let $f : X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a sheaf on Y, and $\mathcal{G}$ a sheaf on X. Then* $\mathrm{Hom}_X(f^{-1}\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_Y(\mathcal{F}, f_*\mathcal{G})$.

Definition 14. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Their **sheaf Hom** is the sheaf $\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$ (or simply $\mathcal{H}om(\mathcal{F}, \mathcal{G})$) defined by

$$\mathcal{H}om_X(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_U(\mathcal{F}|_U, \mathcal{G}|_U).$$

Their **sheaf tensor product** is the sheafification of the presheaf

$$(\mathcal{F} \underset{\mathrm{ps}}{\otimes} \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U).$$

It is denoted $\mathcal{F} \otimes \mathcal{G} = (\mathcal{F} \otimes_{\mathrm{ps}} \mathcal{G})^+$.

Warning 15. It is tempting to define sheaf Hom by setting $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}(U), \mathcal{G}(U))$, but this definition is not correct. Indeed, it does not make sense—there is no way to define restriction maps here, so this "definition" does not even specify a presheaf.

In the context of sheaves, many adjointness theorems come in ordinary and sheaf-theoretic versions. The sheaf-theoretic version of Theorem 13 says:

Theorem 16. $f_* \mathcal{H}om_X(f^{-1}\mathcal{F}, \mathcal{G}) \simeq \mathcal{H}om_Y(\mathcal{F}, f_*\mathcal{G})$.

Here are two more:

Theorem 17. $\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$.

Theorem 18. $\mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H}))$.