Local Systems and Constructible Sheaves

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Convention. Henceforth, all sheaves will be sheaves of complex vector spaces. All topological spaces will be locally compact, Hausdorff, second-countable, locally path-connected, and semilocally simply connected. Unless otherwise specified, they will also be path-connected.

Remark 1. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of complex vector spaces, objects like $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ and $\mathcal{F} \otimes \mathcal{G}$ depend on whether one is working in the category of sheaves of abelian groups, or in the category of sheaves of vector spaces. Indeed, the same phenomenon is already visible with ordinary Hom and $\otimes$: in the category of complex vector spaces, we have $\mathbb{C} \otimes \mathbb{C} \simeq \mathbb{C}$, while in the category of real vector spaces, $\mathbb{C} \otimes \mathbb{C} \simeq \mathbb{R}^4$. In the category of abelian groups, $\mathbb{C} \otimes \mathbb{C}$ is an uncountable-rank free $\mathbb{Z}$-module for which we cannot give an explicit basis.

Henceforth, all Hom-groups, sheaf Hom's, and tensor products are to be computed in the category of sheaves of complex vector spaces.

Definition 2. A sheaf $\mathcal{F}$ on X is **locally constant**, or $\mathcal{F}$ is a **local system**, if for all $x \in X$, there is a neighborhood U containing x such that $\mathcal{F}|_U$ is a constant sheaf.

Example 3. A constant sheaf is locally constant.

Example 4. The square-root sheaf $\mathcal{Q}$ on $\mathbb{C} \smallsetminus \{0\}$ is locally constant, but not constant.

Example 5. Let $\mathcal{F}$ be the sheaf of continuous functions on $X = \mathbb{C}$. This sheaf is *not* locally constant. In general, if U is a connected neighborhood of x, then over any smaller connected neighborhood $V \subset U$ there will be sections (continuous functions) that are not the restriction of any section over U. This situation does not occur in constant sheaves, so $\mathcal{F}|_U$ is not a constant sheaf for any open set U.

Definition 6. A sheaf $\mathcal{F}$ on X is **constructible** if there is a decomposition $X = \bigsqcup_{i=1}^n X_i$ of X into a disjoint union of finitely many locally closed subsets X_i such that $\mathcal{F}|_{X_i}$ is locally constant. (A set is **locally closed** if it is the intersection of an open set and a closed set.)

Note, in particular, that all open sets and all closed sets are locally closed.

Typically, the required decomposition of X will either be obvious or fixed in advance. Proving that a given sheaf is constructible usually consists only of showing that the restrictions $\mathcal{F}|_{X_i}$ are locally constant, and not of finding the decomposition of X.

Theorem 7. *There is a bijection*

$$\left\{\begin{matrix}\text{local systems on } X \\ \text{up to isomorphism}\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{\begin{matrix}\text{representations of } \pi_1(X, x_0) \\ \text{up to isomorphism}\end{matrix}\right\}.$$

In fact, more is true: there is an equivalence of categories between the two sides of this picture.

A subset $K \subset X$ is called *good* (for $\mathcal{F}$ if it is connected, and there is a connected open set V containing K such that $\mathcal{F}|_V$ is a constant sheaf. Note that if K is good, any connected subset $K' \subset K$ is also good.

If $\gamma, \gamma' : [0, 1] \to X$ are two paths in X, we write $\gamma \sim \gamma'$ to indicate that they are homotopic.

Lemma 7.1. Let $x_1, \ldots, x_n$ be points in a good set K. There are natural isomorphisms of stalks $\mathcal{F}_{x_i} \overset{\sim}{\to} \mathcal{F}_{x_j}$ for all i and j. These isomorphisms are compatible with each other: for any i, j, k, the composition $\mathcal{F}_{x_i} \overset{\sim}{\to} \mathcal{F}_{x_j} \overset{\sim}{\to} \mathcal{F}_{x_k}$ coincides with $\mathcal{F}_{x_i} \overset{\sim}{\to} \mathcal{F}_{c_k}$.

Proof. Choose a good open set V containing K. Since $\mathcal{F}|_V$ is constant, the natural maps $\mathcal{F}(V) \to \mathcal{F}_{x_i}$ are all isomorphisms. Composing one of these with the inverse of another gives the desired isomorphisms: $\mathcal{F}_{x_i} \overset{\sim}{\leftarrow} \mathcal{F}(V) \overset{\sim}{\to} \mathcal{F}_{x_j}$. The compatibility is obvious. □

Lemma 7.2. If $\gamma : [0,1] \to X$ (resp. $H : [0,1]^2 \to X$) is a continuous map, there exist numbers $a_0 = 0 < a_1 < \cdots < a_n = 1$ (resp. and $b_0 = 0 < b_1 < \cdots < b_m = 1$) such that for all i, $\gamma([a_i, a_{i+1}])$ (resp. for all i and j, $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$) is good.

Proof. For every point $t \in [0,1]$ (resp. $t \in [0,1]^2$), there is a good open neighborhood of $\gamma(t)$ (resp. $H(t)$), so by continuity, there is an open neighborhood V of t such that $\gamma(V)$ (resp. $H(V)$) is good. Inside V, find an interval $[a, a']$ (resp. box $[a, a'] \times [b, b']$) containing t. The interiors of all these intervals (resp. boxes), as t ranges over all points of $[0,1]$ (resp. $[0,1]^2$), form an open cover. By compactness, we can select finitely of them that still form an open cover. From the remaining intervals (resp. boxes), write all a's and a''s in order as $a_0 < a_1 < \cdots < a_n$ (resp. and write all the b's and b''s as $b_0 < b_1 < \cdots < b_m$). Each interval $[a_i, a_{i+1}]$ (resp. box $[a_i, a_{i+1}] \times [b_j, b_{j+1}]$) is contained in one of the original open sets (called "V" above), so $\gamma([a_i, a_{i+1}])$ (resp. $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$) is good. □

Proof of Theorem 7. The proof proceeds in five steps.

Step A. Given a path $\gamma : [0,1] \to X$, define an invertible linear transformation $\rho(\gamma) : \mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$. Invoke Lemma 7.2 to get a sequence of points $a_0 = 0 < a_1 < \cdots < a_n = 1$. Since $\gamma([a_i, a_{i+1}])$ is good for each i, by invoking Lemma 7.1, we obtain a number of isomorphisms as follows:

$$\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(a_1)} \xrightarrow{\sim} \mathcal{F}_{\gamma(a_2)} \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}.$$

We would like to define $\rho(\gamma)$ to simply be the composition of all of these, but at first glance it is not clear that that would be well-defined: it appears to depend on the a_i's that come out of the invocation of Lemma 7.2.
To deal with this problem, note first that if we add a new point a_{new} to our list, say between a_i and a_{i+1}, then $\gamma([a_i, a_{\text{new}}])$ and $\gamma([a_{\text{new}}, a_{i+1}])$ are good, so we can repeat the above construction. But the total composition $\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$ will remain unchanged, because the triangle

$$\begin{array}{ccc} \mathcal{F}_{\gamma(a_i)} & \xrightarrow{\quad\sim\quad} & \mathcal{F}_{\gamma(a_{i+1})} \\ & \searrow_{\sim} \quad \nearrow_{\sim} & \\ & \mathcal{F}_{\gamma(a_{\text{new}})} & \end{array}$$

commutes by Lemma 7.1. Indeed, by induction adding finitely many points to the a_i's does not change the total composition $\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$.
In general, to show that $\rho(\gamma)$ defined with respect to $a_0, \ldots, a_n$ coincides with the map defined by another set of points, say $a'_0, \ldots, a'_{n'}$, we simply note that both coincide with the map defined with respect to the set of all a_i's and a'_i's. So $\rho(\gamma)$ is well-defined. □
Step B. If $\gamma \sim \gamma'$, then $\rho(\gamma) = \rho(\gamma')$. Let $H : [0,1]^2 \to X$ be a homotopy between γ and γ'. That is, $H(t,0) = \gamma(t)$ and $H(t,1) = \gamma'(t)$. Let $a_0 < a_1 < \cdots < a_n$ and $b_0 < b_1 < \cdots < b_m$ be as given by Lemma 7.2. Let $\gamma_j : [0,1] \to X$ be the path $\gamma_j(t) = H(t, b_j)$. (In particular, $\gamma_0 = \gamma$ and $\gamma_m = \gamma'$.) To prove that $\rho(\gamma) = \rho(\gamma')$, it clearly suffices to prove $\rho(\gamma_j) = \rho(\gamma_{j+1})$ for all j.
Consider the diagram

$$\begin{array}{ccccccccc} \mathcal{F}_{H(0,b_{j+1})} & \xrightarrow{\sim} & \mathcal{F}_{H(a_1,b_{j+1})} & \xrightarrow{\sim} & \mathcal{F}_{H(a_2,b_{j+1})} & \xrightarrow{\sim} & \cdots\cdots & \xrightarrow{\sim} & \mathcal{F}_{H(1,b_{j+1})} \\ \| & & \updownarrow \wr & & \updownarrow \wr & & & & \| \\ \mathcal{F}_{H(0,b_j)} & \xrightarrow{\sim} & \mathcal{F}_{H(a_1,b_j)} & \xrightarrow{\sim} & \mathcal{F}_{H(a_2,b_j)} & \xrightarrow{\sim} & \cdots\cdots & \xrightarrow{\sim} & \mathcal{F}_{H(1,b_j)} \end{array}$$

By Lemma 7.2, every small square in this diagram commutes, because $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$ is good. Since $\rho(\gamma_j)$ is the composition of all the maps along the bottom of this diagram, and $\rho(\gamma')$ is the composition of all maps along the top, we see that $\rho(\gamma) = \rho(\gamma')$. □
Corollary. There is a well-defined map $\rho : \pi_1(X, x_0) \to GL(\mathcal{F}_{x_0})$. This follows immediately from Steps A and B.
Step C. Given a representation $\tau : \pi_1(X, x_0) \to GL(E)$, construct a local system $\mathcal{G}$ on X. For each point $x \in X$, let us choose, once and for all, a path $\alpha_x : [0,1] \to X$ that joins x_0 to x. (That is, $\alpha_x(0) = x_0$ and $\alpha_x(1) = x$.) In particular, let us take α_{x_0} to be the constant path at x_0.

Now, we define a sheaf $\mathcal{G}$ on X by:

$$\mathcal{G}(U) = \left\{ \begin{array}{c|c} \text{functions} & \text{for any path } \gamma : [0,1] \to U, \text{ we have} \\ k : U \to E & k(\gamma(1)) = [\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}] \cdot k(\gamma(0)) \end{array} \right\}$$

Here, "$*$" indicates composition of paths. Note that $\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}$ is a loop based at x_0, so its homotopy class $[\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}]$ is an element of $\pi_1(X, x_0)$. Via τ, this element acts on the vector $k(\gamma(0)) \in E$.

The verification that $\mathcal{G}$ is sheaf is routine, and we omit it. It does, however, remain to show that $\mathcal{G}$ is locally constant. Here we require the fact that X is semilocally simply connected. That is, every point has a neighborhood V with the property that every loop in V is null-homotopic in X.

Given such a V, we will now show that $\mathcal{G}(V) \simeq E$. Choose a point $x \in V$, and define a map $\phi : \mathcal{G}(V) \to E$ by $\phi(k) = k(x)$. It is easy to check that ϕ is injective. Indeed, a section $k \in \mathcal{G}(V)$ is determined by the vector $k(x) \in E$ as follows: for any $y \in V$, we have $k(y) = [\alpha_y^{-1} * \gamma * \alpha_x] \cdot k(x)$, where γ is any path joining x to y. To show that ϕ is surjective, given $e \in E$, we define a function $k \in \mathcal{G}(V)$ by

$$k(y) = [\alpha_y^{-1} * \gamma * \alpha_x] \cdot e \qquad \text{where } \gamma \text{ is a path joining } x \text{ to } y.$$

(In particular, if $y = x$, we take γ to be the constant path at x, so that $\phi(k) = k(x)$ is indeed e.) But now there is a well-definedness issue arising from the choice of γ. For a given $y \in V$, suppose γ and γ' are two different paths joining x to y. Then $\gamma^{-1} * \gamma'$ is a loop in V, which is, by assumption, null-homotopic in X. It follows from this that $\gamma \sim \gamma'$ in X, and therefore that $[\alpha_y^{-1} * \gamma * \alpha_x] = [\alpha_y^{-1} * \gamma' * \alpha(x)] \in \pi_1(X, x_0)$. Thus, $k(y)$ is independent of the choice of γ, and k is well-defined. We conclude that $\phi : \mathcal{G}(V) \xrightarrow{\sim} E$ is an isomorphism.

Now, the same argument can be repeated for any connected open set $V' \subset V$, so $\mathcal{G}(V') \simeq E$ for such sets as well. It follows then that in fact $\mathcal{G}|_V \simeq \underline{E}_V$. Since a neighborhood V of this type can be found around every point in X, we conclude that $\mathcal{G}$ is locally constant. □

At this point, we have a construction $\mathcal{F} \mapsto \rho$ assigning a representation to any local system, and another, $\tau \mapsto \mathcal{G}$, assigning a local system to any representation. It remains to show that these two assignments are inverses of one another.

Step D. Given $\mathcal{F}$ and ρ as in Steps A–B, take $\tau = \rho$ and construct $\mathcal{G}$ as in Step C. Then $\mathcal{F} \simeq \mathcal{G}$. Note that the vector space E of Step C is now identified with $\mathcal{F}_{x_0}$. We will actually work with the sheafification $\mathcal{F}^+$ of $\mathcal{F}$ rather than $\mathcal{F}$ itself. (Of course, since $\mathcal{F}$ is already a sheaf, the two are isomorphic.) First, we define a morphism $\phi : \mathcal{F}^+ \to \mathcal{G}$. Let U be a connected open set such that $\mathcal{F}^+|_U$ is a constant sheaf. We define $\phi_U : \mathcal{F}^+(U) \to \mathcal{G}(U)$ as follows: given $s \in \mathcal{F}^+(U)$ (recall that s is a function $U \to \coprod_{x \in U} \mathcal{F}_x$), we define $\phi_U(s) \in \mathcal{G}(U)$ to be the function

$$\phi_U(s) : U \to \mathcal{F}_{x_0}, \qquad \phi_U(s)(x) = \rho(\alpha_x^{-1}) \cdot s(x).$$

(This makes sense: $s(x) \in \mathcal{F}_x$, and α_x^{-1} is a path from x to x_0, so $\rho(\alpha_x^{-1})$ is a linear transformation $\mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_{x_0}$.) We need to check that $\phi_U(s)$ is a valid section of $\mathcal{G}$: given a path $\gamma : [0,1] \to U$, note first that $\gamma([0,1])$ is good, so the linear transformation $\rho(\gamma)$ constructed in Step A coincides with the canonical isomorphism $\mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$ of Lemma 7.1. The latter is defined by taking germs of a given section, so we see that $\rho(\gamma) \cdot s(\gamma(0)) = s(\gamma(1))$. Then,

$$\begin{aligned} \phi_U(s)(\gamma(1)) &= \rho(\alpha_{\gamma(1)}^{-1}) \cdot s(\gamma(1)) = \rho(\alpha_{\gamma(1)}^{-1})\rho(\gamma) \cdot s(\gamma(0)) \\ &= \rho(\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)})\rho(\alpha_{\gamma(0)}^{-1}) \cdot s(\gamma(0)) = \rho([\alpha_{\gamma(1)}^{-1} * \gamma * \alpha_{\gamma(0)}]) \cdot \phi_U(s)(\gamma(0)). \end{aligned}$$

So $\phi_U(s)$ is indeed a valid section of $\mathcal{G}(U)$.

Although we have only defined ϕ_U for certain open sets U, those sets suffice—ϕ_U is determined on other open sets by gluing. The verification of this is left as an exercise.

Next, we define a morphism $\psi : \mathcal{G} \to \mathcal{F}^+$. This time, let V be an open set in which all loops are null-homotopic in X (again, we are using the semilocally-simply-connectedness of X). Given $k \in \mathcal{G}(V)$ (a certain function $k : U \to \mathcal{F}_{x_0}$), define $\psi_V(k) \in \mathcal{F}^+(V)$ by

$$\psi_V(k) : U \to \coprod_{x \in V} \mathcal{F}_x, \qquad \psi_V(k)(x) = \rho(\alpha_x) \cdot k(x).$$

This time, we must check the local condition for $\psi_V(k)$ to be a valid section of $\mathcal{F}^+(V)$. Let $U \subset V$ be a connected neighborhood of x such that $\mathcal{F}|_U$ is constant, and let $s \in \mathcal{F}(U)$ be such that $s_x = \psi_V(k)(x)$. We will show that for all $y \in U$, $s_y = \psi_V(k)(y)$. As argued above, $s_y = \rho(\gamma) \cdot s_x$, where γ is any path in U joining x to y. Recall also that k satsifies $k(y) = \rho(\alpha_y^{-1} * \gamma * \alpha_x) \cdot k(x)$. Now,

$$\begin{aligned} \psi_V(k)(y) &= \rho(\alpha_y) \cdot k(y) = \rho(\alpha_y) \cdot \rho(\alpha_y^{-1} * \gamma * \alpha_x) \cdot k(x) \\ &= \rho(\gamma) \cdot \rho(\alpha_x) \cdot k(x) = \rho(\gamma) \cdot s_x = s_y. \end{aligned}$$

As before, it suffices to define ψ on a certain collection of open sets.

It remains to check that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity morphisms of $\mathcal{F}^+$ and $\mathcal{G}$, respectively. This is straightforward from the formulas. □

Step E. Given τ and $\mathcal{G}$ as in Step C, take $\mathcal{F} = \mathcal{G}$ and construct ρ as in Steps A–B. Then $\rho = \tau$. Note first that all stalks of $\mathcal{G}$ are copies of E, and the germ of a section k at x is simply its value $k(x)$. Let U be an open set such that $\mathcal{G}|_U$ is a constant sheaf, and let $\gamma : [0,1] \to U$ be a path in U. Following the proof of Lemma 7.1, we construct $\rho(\gamma) : E \to E$ as follows: given $e \in E$, take a function $k \in \mathcal{G}(U)$ such that $k(x) = e$; then $\rho(\gamma) \cdot e = k(y)$. We have $k(y) = \tau([\alpha_y^{-1} * \gamma * \alpha_x]) \cdot k(x)$, so $\rho(\gamma) = \tau([\alpha_y^{-1} * \gamma * \alpha_x])$.

Now, if γ is a loop based at x_0, let us follow the construction of $\rho(\gamma)$ in Step A. Take $a_0 = 0 < a_1 < \cdots < a_n = 1$ such that $\gamma([a_i, a_{i+1}])$ is good. The action of each restricted path $\gamma|_{[a_i,a_{i+1}]}$ is constructed as in the previous paragraph, and $\rho(\gamma)$ is the composition of all of these. Thus:

$$\begin{aligned} \rho(\gamma) &= \tau([\alpha_{x_0}^{-1} * \gamma|_{[a_{n-1},1]} * \alpha_{\gamma(a_{n-1})}]) \cdot \tau([\alpha_{\gamma(a_{n-1})}^{-1} * \gamma|_{[a_{n-2},a_{n-1}]} * \alpha_{\gamma(a_{n-2})}]) \cdots \tau([\alpha_{\gamma(a_1)}^{-1} * \gamma|_{[0,a_1]} * \alpha_{x_0}]) \\ &= \tau([\gamma|_{[a_{n-1},1]} * \gamma|_{[a_{n-2},a_{n-1}]} * \cdots * \gamma_{[0,a_1]}]) = \tau([\gamma]). \end{aligned}$$

Here we have used the fact that α_{x_0} is the constant path. This completes the proof of the theorem. □