Categories of Complexes; Distinguished Triangles

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Definition 1. An **abelian category** is a category satisfying the following four axioms:

(1) For any two objects A and B in $\mathcal{A}$, the set of morphisms $\mathrm{Hom}(A, B)$ is endowed with the structure of an abelian group, and composition of morphisms is biadditive.
(2) There is a "zero object" 0 with the property that $\mathrm{Hom}(0, 0) = 0$. (This property implies that it is unique up to unique isomorphism, and that $\mathrm{Hom}(0, A) = \mathrm{Hom}(A, 0) = 0$ for all other objects A.)
(3) For any two objects A and B, there is a "biproduct" $A \oplus B$. It is equipped with morphisms as shown below:

$$\begin{tikzcd} A \arrow[r, shift left, "i_1"] & A \oplus B \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & B \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

These morphisms satisfy the following identities:

$$p_1 \circ i_1 = \mathrm{id}_A \qquad p_1 \circ i_2 = 0 \qquad i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_{A \oplus B}$$
$$p_2 \circ i_2 = \mathrm{id}_B \qquad p_2 \circ i_1 = 0$$

(4) Every morphism $\phi : A \to B$ gives rise to a diagram

$$\begin{tikzcd} K \arrow[r, "k"] & A \arrow[r, "i"'] \arrow[rr, bend left, "\phi"] & I \arrow[r, "j"'] & B \arrow[r, "c"] & C \end{tikzcd}$$

Here, $j \circ i = \phi$. Also, K is the kernel of ϕ, C is its cokernel, and I is both the cokkernel of $k : K \to A$ and the kernel of $c : B \to C$. Such a diagram is called the **canonical decomposition** of ϕ. The object I is called the **image** of ϕ.

A category satisfying the first three of these axioms is called an **additive category**.

Definition 2. The **category of complexes** over $\mathcal{A}$, denoted $C(\mathcal{A})$, is the category whose objects are **complexes** of objects of $\mathcal{A}$: that is, a sequence of objects (A^i) labelled by integers, together with morphisms (called **differentials**) $d^i : A^i \to A^{i+1}$, such that $d^i \circ d^{i-1} = 0$.

$$A^\bullet : \quad \cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \quad \text{where } d^i \circ d^{i-1} = 0 \text{ for all } i,$$

in which a morphism $f : A^\bullet \to B^\bullet$ is a family of morphisms in $\mathcal{A}$, $f^i : A^i \to B^i$, that are compatible with the differentials. That is, every square in the following diagram should commute:

$$\begin{tikzcd} \cdots \arrow[r] & A^{-1} \arrow[r] \arrow[d, "f^{-1}"'] & A^0 \arrow[r] \arrow[d, "f^0"'] & A^1 \arrow[r] \arrow[d, "f^1"'] & \cdots \\ \cdots \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots \end{tikzcd}$$

Definition 3. The ith **cohomology functor** $H^i : C(\mathcal{A}) \to \mathcal{A}$ is defined as follows: given an object $A^\bullet$ in $C(\mathcal{A})$, let I be image of d^{i-1}, and K the kernel of d^i. The natural morphism $I \to A^i$ factors through K (by the universal property of the kernel, and the fact that $d^i \circ d^{i-1} = 0$). $H^i(A^\bullet)$ is defined to be the cokernel of the induced morphism $I \to K$. (In short, $H^i(A^\bullet) = \ker d^i / \operatorname{im} d^{i-1}$.) It is easy to check that a morphism $f : A^\bullet \to B^\bullet$ induces a morphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$, so H^i is indeed a functor.

Definition 4. A **homotopy** between two morphisms $f, g : A^\bullet \to B^\bullet$ in $C(\mathcal{A})$ is a collection of morphisms $h^i : A^i \to B^{i-1}$, such that

$$dh + hd = f - g.$$

That is, in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r] \arrow[dl, "h^{-1}"'] \arrow[d, shift right, "f^{-1}"'] \arrow[d, shift left, "g^{-1}"] & A^{0} \arrow[r] \arrow[dl, "h^{0}"'] \arrow[d, shift right, "f^{0}"'] \arrow[d, shift left, "g^{0}"] & A^{1} \arrow[r] \arrow[dl, "h^{1}"'] \arrow[d, shift right, "f^{1}"'] \arrow[d, shift left, "g^{1}"] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r] & B^{0} \arrow[r] & B^{1} \arrow[r] & \cdots
\end{tikzcd}$$

we have $d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i = f^i - g^i$ for each i. If there is a homotopy between f and g, then f and g are said to be **homotopic**.

Definition 5. The **homotopy category** over $\mathcal{A}$, denoted $K(\mathcal{A})$, is the category whose objects are complexes of objects in $\mathcal{A}$, but whose morphisms are given by

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet)/(\text{morphisms homotopic to } 0).$$

In other words, a morphism in $K(\mathcal{A})$ is a homotopy class of morphisms in $C(\mathcal{A})$.

Lemma 6. *If $f, g : A^\bullet \to B^\bullet$ are two homotopic morphisms in $C(\mathcal{A})$, then $H^i(f) = H^i(g)$.*

Corollary 7. *The cohomology functors can be regarded as being functors $K(\mathcal{A}) \to \mathcal{A}$.*

Definition 8. A morphism $f : A^\bullet \to B^\bullet$ (in $C(\mathcal{A})$ or $K(\mathcal{A})$) is a **quasi-isomorphism** (often abbreviated to **qis**) if the morphisms $H^i(f)$ in $\mathcal{A}$ are isomorphisms for all i.

Definition 9. The **derived category** of $\mathcal{A}$, denoted $D(\mathcal{A})$, is the category whose objects are complexes of objects in $\mathcal{A}$, and in which a morphism from $A^\bullet$ to $B^\bullet$ is an equivalence class of "roofs"

$$\begin{tikzcd}
 & X^\bullet \arrow[dl, "q"'] \arrow[dr, "u"] \arrow[d, phantom, "\text{qis}"] & \\
A^\bullet & {} & B^\bullet
\end{tikzcd}$$

where q is a quasi-isomorphism. The equivalence relation is as follows: Two roofs

$$\begin{tikzcd}
 & X^\bullet \arrow[dl, "q"'] \arrow[dr, "u"] \arrow[d, phantom, "\text{qis}"] & \\
A^\bullet & {} & B^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & Y^\bullet \arrow[dl, "r"'] \arrow[dr, "v"] \arrow[d, phantom, "\text{qis}"] & \\
A^\bullet & {} & B^\bullet
\end{tikzcd}$$

are equivalent if there exists a roof over $X^\bullet$ and $Y^\bullet$ making the following diagram commute:

$$\begin{tikzcd}
 & & Z^\bullet \arrow[dl] \arrow[dr] & & \\
 & X^\bullet \arrow[dl] \arrow[drrr] & & Y^\bullet \arrow[dr] \arrow[dlll] & \\
A^\bullet & & & & B^\bullet
\end{tikzcd}$$

Definition 10. Let $n \in \mathbb{Z}$, and let $A^\bullet$ be a complex. The nth **translation** (or **shift**) is the complex $A[n]^\bullet$ defined by

$$A[n]^i = A^{i+n} \qquad \text{with differential} \qquad d^i = (-1)^n d_A^{i+n}.$$

Given a morphism $f : A^\bullet \to B^\bullet$, the translated morphism $f[n] : A[n]^\bullet \to B[n]^\bullet$ is defined by $f[n]^i = f^{i+n}$. In this way, translation by n is a functor $C(\mathcal{A}) \to C(\mathcal{A})$ (or $K(\mathcal{A}) \to K(\mathcal{A})$, or $D(\mathcal{A}) \to D(\mathcal{A})$).

Definition 11. Let $f : A^\bullet \to B^\bullet$ be a morphism in $C(\mathcal{A})$. The **cone** of f a complex, denoted $\mathrm{cone}^\bullet f$, defined as follows:

$$\mathrm{cone}^i f = A^{i+1} \oplus B^i \qquad \text{with differential} \qquad \begin{array}{c} d^i : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1} \\ \text{given by } \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} \end{array}$$

It is easy to check that $d^{i+1} \circ d^i = 0$:

$$d^{i+1} \circ d^i = \begin{pmatrix} -d_A^{i+2} & 0 \\ f^{i+2} & d_B^{i+1} \end{pmatrix} \cdot \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} = \begin{pmatrix} d_A^{i+2} \circ d_A^{i+1} & 0 \\ -f^{i+2} \circ d_A^{i+1} + d_B^{i+1} \circ f^{i+1} & d_B^{i+1} \circ d_B^i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

There are natural morphisms $B^\bullet \to \operatorname{cone}^\bullet f$ and $\operatorname{cone}^\bullet f \to A[1]^\bullet$:

$$B^\bullet \xrightarrow{\binom{0}{\mathrm{id}}} \operatorname{cone}^\bullet f \xrightarrow{(\mathrm{id}\ 0)} A[1]^\bullet$$

Definition 12. A sequence of morphisms $X^\bullet \to Y^\bullet \to Z^\bullet \to Z[1]^\bullet$ in $K(\mathcal{A})$ or $D(\mathcal{A})$ is called a **distinguished triangle** (abbreviated to **d.t.**) if there is a commutative diagram

$$\begin{tikzcd}
X^\bullet \arrow[r] \arrow[d, "q"'] & Y^\bullet \arrow[r] \arrow[d, "r"'] & Z^\bullet \arrow[r] \arrow[d, "s"'] & X[1]^\bullet \arrow[d, "q[1]"] \\
A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r] & \operatorname{cone}^\bullet f \arrow[r] & A[1]^\bullet
\end{tikzcd}$$

where q, r, and s are isomorphisms. (Of course, the meaning of "isomorphism," and hence of "distinguished triangle," depends on whether one is working in $K(\mathcal{A})$ or $D(\mathcal{A})$.)

Theorem 13. *Distinguished triangles in $K(\mathcal{A})$ enjoy the following four properties:*

(1) **(Identity)** The triangle $A^\bullet \xrightarrow{\mathrm{id}} A^\bullet \to 0 \to A[1]^\bullet$ is distinguished.
(2) **(Rotation)** The triangle $A^\bullet \xrightarrow{f} B^\bullet \to C^\bullet \to A[1]^\bullet$ is distinguished if and only if the **rotated** triangle $B^\bullet \to C^\bullet \to A[1]^\bullet \xrightarrow{-f[1]} B[1]^\bullet$ is.
(3) **(Square Completion)** Any commutative square

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "u"'] & B^\bullet \arrow[d] \\
F^\bullet \arrow[r] & G^\bullet
\end{tikzcd}$$

can be completed to a commutative diagram of distinguished triangles:

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "r"'] & B^\bullet \arrow[r, dashed] \arrow[d] & C^\bullet \arrow[r, dashed] \arrow[d, dashed] & A[1]^\bullet \arrow[d, "r[1]"] \\
F^\bullet \arrow[r] & G^\bullet \arrow[r, dashed] & H^\bullet \arrow[r, dashed] & F[1]^\bullet
\end{tikzcd}$$

(4) **(Octahedral Property)** Given a commutative diagram of morphisms

$$\begin{tikzcd}
& B^\bullet \arrow[dr, "g"] & \\
A^\bullet \arrow[ur, "f"] \arrow[rr, "h=g\circ f"'] & & C^\bullet
\end{tikzcd},$$

extend each of f, g, and h to a distinguished triangle:

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow F^\bullet \longrightarrow A[1]^\bullet$$

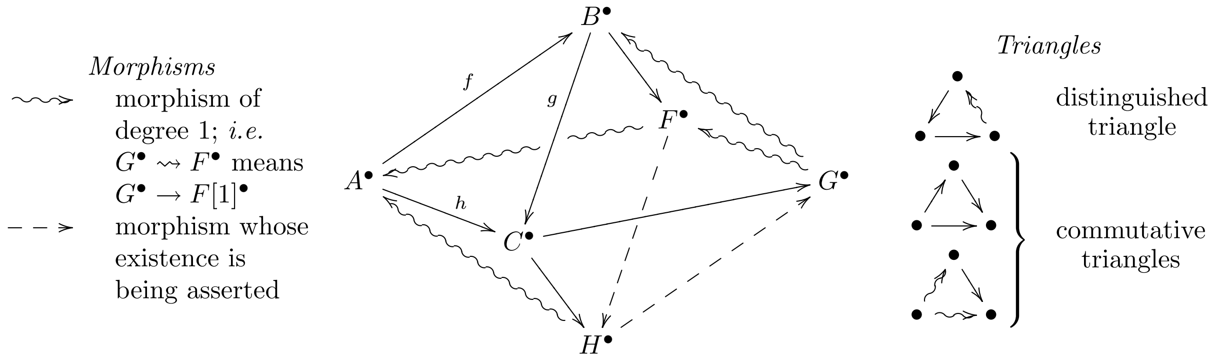
$$B^\bullet \xrightarrow{g} C^\bullet \longrightarrow G^\bullet \longrightarrow B[1]^\bullet$$

$$A^\bullet \xrightarrow{h} C^\bullet \longrightarrow H^\bullet \longrightarrow A[1]^\bullet$$

Arrange three triangles, together with the obvious morphism $G^\bullet \to F[1]^\bullet$ (composition of $G^\bullet \to B[1]^\bullet$ and $B[1]^\bullet \to F[1]^\bullet$) can be arranged in an octahedron as shown below. There exist morphisms $F^\bullet \to H^\bullet$ and $H^\bullet \to G^\bullet$ making

$$F^\bullet \longrightarrow H^\bullet \longrightarrow G^\bullet \longrightarrow F[1]^\bullet$$

a distinguished triangle, and making the neighboring faces of the octahedron commute.



Remark 14. This theorem essentially states that $K(\mathcal{A})$ satisfies the axioms for a **triangulated category**.

Proof. (1) The following diagram is clearly commutative:

$$\begin{tikzcd}
A^\bullet \arrow[r, "\mathrm{id}"] \arrow[d, equal] & A^\bullet \arrow[r] \arrow[d, equal] & \operatorname{cone}^\bullet \mathrm{id} \arrow[r] \arrow[d] & A[1]^\bullet \arrow[d, equal] \\
A^\bullet \arrow[r, "\mathrm{id}"] & A^\bullet \arrow[r] & 0 \arrow[r] & A[1]^\bullet
\end{tikzcd}$$

It remains to check that $\operatorname{cone}^\bullet \mathrm{id} \to 0$ is an isomorphism in $K(\mathcal{A})$. Its putative inverse, of course, is the zero morphism $0 \to \operatorname{cone}^\bullet \mathrm{id}$; we need to check that the compositions in both directions are equal (in $K(\mathcal{A})$, *i.e.*, homotopic) to the identity morphisms of the their respective objects. For the zero object, this is trivial. For $\operatorname{cone}^\bullet \mathrm{id}$, we now show that the zero morphism is homotopic to the identity morphism. Let $h^i : \operatorname{cone}^i \mathrm{id} \to \operatorname{cone}^{i-1} \mathrm{id}$ be the homotopy given by

$$h^i : A^{i+1} \oplus A^i \to A^i \oplus A^{i-1}, \qquad h^i = \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}$$

Then

$$dh + hd = \begin{pmatrix} -d_A & 0 \\ \mathrm{id} & d_A \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_A & 0 \\ \mathrm{id} & d_A \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \mathrm{id}_{\operatorname{cone}^\bullet \mathrm{id}} - 0.$$

Thus, $\operatorname{cone}^\bullet \mathrm{id} \to 0$ is an isomorphism. ☐

(2) Starting with a distinguished triangle $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} \operatorname{cone}^\bullet f \xrightarrow{r} A[1]^\bullet$, we must show that the rotated triangle is isomorphic to the distinguished triangle obtained by taking the cone of g. Note that $\operatorname{cone}^i f = A^{i+1} \oplus B^i$, and $\operatorname{cone}^i g = B^{i+1} \oplus \operatorname{cone}^i f = B^{i+1} \oplus A^{i+1} \oplus B^i$. Consider the following diagram:

$$\begin{tikzcd}
B^\bullet \arrow[r, "g"] \arrow[d, equal] & \operatorname{cone}^\bullet f \arrow[r, "r"] \arrow[d, equal] & A[1]^\bullet \arrow[r, "-f[1]"] \arrow[d, "\theta"] & B[1]^\bullet \arrow[d, equal] \\
B^\bullet \arrow[r, "g"] & \operatorname{cone}^\bullet f \arrow[r, "s"] & \operatorname{cone}^\bullet g \arrow[r, "t"] & B[1]^\bullet
\end{tikzcd}$$

Here, g, r, s and t are the usual morphisms, and θ is defined as below:

$$g = \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix}, \qquad r = \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}, \qquad t = \begin{pmatrix} \mathrm{id} & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \theta = \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix}$$

For future reference, we note that the differentials are given by

$$d_{\operatorname{cone}^\bullet f} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \qquad \text{and} \qquad d_{\operatorname{cone}^\bullet g} = \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix}$$

We must check first that the diagram is commutative, and then that θ is an isomorphism. It is clear that $t\theta = \mathrm{id} \circ -f[1]$, but not that $\theta r = s \circ \mathrm{id}$. For the latter, note that

$$s - \theta r = \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix} - \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

We define a homotopy $h^i : \mathrm{cone}^i f \to \mathrm{cone}^{i-1} g$ by

$$h = \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$dh + hd = \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix} = s - \theta r.$$

Thus, s is homotopic to θr, so the diagram is commutative.

To show that θ is an isomorphism, define $\psi : \mathrm{cone}^\bullet g \to A[1]^\bullet$ by the matrix $\begin{pmatrix} 0 & \mathrm{id} & 0 \end{pmatrix}$. It is obvious that $\psi\theta = \mathrm{id}_{A[1]^\bullet}$. To show that $\theta\psi = \mathrm{id}_{\mathrm{cone}^\bullet g}$, we need the homotopy

$$k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$\mathrm{id}_{\mathrm{cone}^\bullet g} - \theta\psi = \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} - \begin{pmatrix} 0 & -f & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Then

$$\begin{aligned} dk + kd &= \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d_B & 0 & 0 \\ 0 & -d_A & 0 \\ \mathrm{id} & f & d_B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -d_B \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} + \begin{pmatrix} \mathrm{id} & f & d_B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & f & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} = \mathrm{id}_{\mathrm{cone}^\bullet g} - \theta\psi. \end{aligned}$$

Thus, θ is an isomorphism, and the rotated triangle is distinguished.

(3) This is the easiest property to prove. We need to define $\theta : \mathrm{cone}^\bullet f \to \mathrm{cone}^\bullet g$ such that the following diagram commutes:

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "r"'] & B^\bullet \arrow[r] \arrow[d, "s"'] & \mathrm{cone}^\bullet f \arrow[r] \arrow[d, "\theta"'] & A[1]^\bullet \arrow[d, "r[1]"] \\ F^\bullet \arrow[r, "g"] & G^\bullet \arrow[r] & \mathrm{cone}^\bullet g \arrow[r] & F[1]^\bullet \end{tikzcd}$$

Define $\theta^i : A^{i+1} \oplus B^i \to F^{i+1} \oplus G^i$ by the matrix

$$\theta = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}.$$

Using the fact that $sf = gr$, it is trivial to check that θ commutes with differentials (so that it is actually a well-defined morphism) and that it makes the above diagram commute.

(4) This, of course, is the hardest property to check, but only in terms of the number of calculations to be done. Assume that $F^\bullet = \text{cone}^\bullet f$, $G^\bullet = \text{cone}^\bullet g$, and $H^\bullet = \text{cone}^\bullet h$. Recall:

$$F^i = A^{i+1} \oplus B^i \qquad G^i = B^{i+1} \oplus C^i \qquad H^i = A^{i+1} \oplus C^i$$

$$d_F = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} \qquad d_G = \begin{pmatrix} -d_B & 0 \\ g & d_C \end{pmatrix} \qquad d_H = \begin{pmatrix} -d_A & 0 \\ h & d_C \end{pmatrix}$$

Define the following morphisms:

$$r : F^\bullet \to H^\bullet \qquad s : H^\bullet \to G^\bullet \qquad t : G^\bullet \to F[1]^\bullet$$

$$r = \begin{pmatrix} \text{id} & 0 \\ 0 & g \end{pmatrix} \qquad s = \begin{pmatrix} f & 0 \\ 0 & \text{id} \end{pmatrix} \qquad t = \begin{pmatrix} 0 & 0 \\ \text{id} & 0 \end{pmatrix}$$

Next, let $K^\bullet = \text{cone}^\bullet r$, so that

$$K^i = F^{i+1} \oplus H^i = A^{i+1} \oplus B^{i+1} \oplus A^{i+1} \oplus C^i.$$

The differential of $K^\bullet$ and the natural maps $u : H^\bullet \to K^\bullet$ and $v : k^\bullet \to F[1]]^\bullet$ are given by:

$$d_K = \begin{pmatrix} d_A & 0 & 0 & 0 \\ -f & -d_B & 0 & 0 \\ \text{id} & 0 & -d_A & 0 \\ 0 & g & h & d_C \end{pmatrix} \qquad u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \qquad v = \begin{pmatrix} \text{id} & 0 & 0 & 0 \\ 0 & \text{id} & 0 & 0 \end{pmatrix}$$

Next, define $\theta : G^\bullet \to K^\bullet$ and $\psi : K^\bullet \to G^\bullet$ by

$$\theta = \begin{pmatrix} 0 & 0 \\ \text{id} & 0 \\ 0 & 0 \\ 0 & \text{id} \end{pmatrix} \qquad \psi = \begin{pmatrix} 0 & \text{id} & f & 0 \\ 0 & 0 & 0 & \text{id} \end{pmatrix}$$

Finally, define homotopies $k : H^i \to K^{i-1}$ and $l : K^i \to K^{i-1}$ by

$$k = \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad l = \begin{pmatrix} 0 & 0 & \text{id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We need to show that the following diagram is well-defined and commutative to establish that the top row is a distinguished triangle:

$$\begin{tikzcd}
F^\bullet \arrow[r, "r"] \arrow[d, equal] & H^\bullet \arrow[r, "s"] \arrow[d, equal] & G^\bullet \arrow[r, "t"] \arrow[d, "\theta"] & F[1] \arrow[d, equal] \\
F^\bullet \arrow[r, "r"] & H^\bullet \arrow[r, "u"] & K^\bullet \arrow[r, "v"] \arrow[u, bend left, "\psi"] & F[1]^\bullet
\end{tikzcd}$$

To establish that this diagram makes sense, the following assertions need to be checked:

- r, s, t, θ, and ψ are well-defined (they commute with differentials).
- $\theta s = u$ (in fact, they are homotopic, and $dk + kd = u - \theta f$).
- θ is an isomorphism (in fact, $\psi\theta = \text{id}_G$, and $\theta\psi$ is homotopic to id_K: we have $dl + ld = \text{id}_K - \theta\psi$).

Finally, to check that this distinguished triangle is compatible with the octahedron, we must check:

- t coincides with the composition $G^\bullet \to B[1]^\bullet \to F[1]^\bullet$.
- The natural morphism $F^\bullet \to A[1]^\bullet$ coincides with the composition $F^\bullet \xrightarrow{r} H^\bullet \to A[1]^\bullet$.
- The natural morphism $C^\bullet \to G^\bullet$ coincides with the composition $C^\bullet \to H^\bullet \xrightarrow{s} G^\bullet$.

The proofs of all these assertions are straightforward matrix calculations. □