Derived Categories and Derived Functors

March 6, 2007

Recall that distinguished triangles in $K(\mathcal{A})$ have four important properties. These properties are essential for even showing that $D(\mathcal{A})$ is a well-defined category: specifically, composition of morphisms in $D(\mathcal{A})$ is not an obvious operation, and we have to use distinguished triangles in $K(\mathcal{A})$ to show how to do it.

Theorem 1. *Distinguished triangles in $D(\mathcal{A})$ enjoy the following properties:*

(0) ***(Existence)*** *Any morphism $A^\bullet \to B^\bullet$ can be completed to a distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$.*

(1–4) *The Identity, Rotation, Completion, and Octahedral properties of distinguished triangles in $K(\mathcal{A})$.*

Proposition 2. *A distinguished triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ in $K(\mathcal{A})$ or $D(\mathcal{A})$ gives rise to a long exact sequence in cohomology*

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots .$$

The rest of this set of notes is devoted to discussing how a functor of abelian categories gives rise to a functor of derived categories. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then:

(1) F induces a functor of chain complexes $F : C(\mathcal{A}) \to C(\mathcal{B})$.

(2) F induces a functor of homotopy categories $F : K(\mathcal{A}) \to K(\mathcal{B})$. (The proof of this is just to check that $F : C(\mathcal{A}) \to C(\mathcal{B})$ takes homotopic morphisms to homotopic morphisms.)

Now, suppose F is actually an exact functor. Then one also has:

(3) F induces a functor of derived categories $F : D(\mathcal{A}) \to D(\mathcal{B})$. The proof requires showing that $F : K(\mathcal{A}) \to K(\mathcal{B})$ takes quasi-isomorphisms to quasi-isomorphisms; that fact is not true in general if F is not exact.

(4) $F : D(\mathcal{A}) \to D(\mathcal{B})$ takes distinguished triangles to distinguished triangles.

The harder, and more interesting, problem is that of defining a functor of derived categories when F is *not* exact. In this case, the constructions requires the concepts of *resolutions* and *adapted classes*.

Definition 3. The **bounded-below category of complexes** (resp. **bounded-below homotopy category**, **bounded-below derived category**) over $\mathcal{A}$ is the full subcategory $C^+(\mathcal{A})$ (resp. $K^+(\mathcal{A})$, $D^+(\mathcal{A})$) of $C(\mathcal{A})$ (resp. $K(\mathcal{A})$, $D(\mathcal{A})$) containing those objects $A^\bullet$ for which there exists an integer N such that $H^i(A^\bullet) = 0$ for all $i < N$. (Here, N may depend on the object $A^\bullet$.) The **bounded-above** categories $C^-(\mathcal{A})$, $K^-(\mathcal{A})$, $D^-(\mathcal{A})$ are defined similarly.

Definition 4. A class of objects $\mathcal{R}$ in $\mathcal{A}$ is **large enough** (or one says that $\mathcal{A}$ **has enough objects in** $\mathcal{R}$) if for every object A of $\mathcal{A}$, there is a monomorphism $A \to R$, where $R \in \mathcal{R}$.

Theorem 5 (Existence of Resolutions)**.** *If $\mathcal{R}$ is large enough, then for any complex $A^\bullet \in D^+(\mathcal{A})$, there exists a complex $R^\bullet \in D^+(\mathcal{A})$ with $R^i \in \mathcal{R}$ for all i, and a quasi-isomorphism $t : A^\bullet \to R^\bullet$.*

Definition 6. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. A class of objects $\mathcal{R}$ in $\mathcal{A}$ is an **adapted class of objects** for F if the following two conditions are satisfied:

(1) F is exact on $\mathcal{R}$ (that is, F takes any short exact sequence of objects in $\mathcal{R}$ to a short exact sequence in $\mathcal{B}$).

(2) $\mathcal{R}$ is large enough.

Definition 7 (Derived Functor)**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a derived functor, and let $\mathcal{R}$ be an adapted class for F. The **derived functor** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is defined as follows: for any object $A^\bullet$ in $D^+(\mathcal{A})$, choose an $\mathcal{R}$-resolution $R^\bullet$, and let $RF(A^\bullet)$ be the complex $F(R^\bullet)$. The latter is well-defined up to quasi-isomorphism (*i.e.*, up to isomorphism in $D(\mathcal{B})$) because $R^\bullet$ is unique up to quasi-isomorphism, and since F is exact on $\mathcal{R}$, it takes quasi-isomorphisms of complexes of objects in $\mathcal{R}$ to quasi-isomorphisms.

There is a further issue of well-definedness: the choice of the adapted class $\mathcal{R}$. That is, if $\mathcal{R}$ and $\mathcal{R}'$ are two adapted classes for F, do they both give rise to the same notion of derived functor? It can be shown that the answer is yes, *i.e.*, that derived functors are independent of the choice of adapted class, but we will not treat this question in full generality. Instead, there is a special case in which the answer is obviously yes, and this case covers all the examples we will meet. Specifically, if it happens that $\mathcal{R} \subset \mathcal{R}'$, then every $\mathcal{R}$-resolution is also an $\mathcal{R}'$-resolution, so the two RF's defined with respect to these two classes coincide.

In our examples, we will only work with adapted classes that contain the class of injective objects.

Definition 8. An object I of an abelian category $\mathcal{A}$ is **injective** if, for every morphism $f : A \to I$ and monomorphism $i : A \hookrightarrow B$, there exists a morphism $g : B \to I$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[dr, "f"'] & B \arrow[d, dashed, "g"] \\ & I \end{tikzcd}$$

Lemma 9. *Any left-exact functor is exact on the class $\mathcal{I}$ of injective objects.*

Corollary 10. *If $\mathcal{A}$ has enough injectives, then the class $\mathcal{I}$ of injective objects is an adapted class for every left-exact functor $F : \mathcal{A} \to \mathcal{B}$.*

Definition 11. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor. The ith **classical derived functor** $R^iF : \mathcal{A} \to \mathcal{B}$ is defined as follows: given an object $A \in \mathcal{A}$, regard it as a complex in $D(\mathcal{A})$ with a single nonzero term located in degree 0. Then $R^iF(A) = H^i(RF(A))$.

Example 12. Let C be an abelian group. The functor $\mathrm{Hom}(C, -) : \mathfrak{Ab} \to \mathfrak{Ab}$ is left-exact, and the category $\mathfrak{Ab}$ has enough injectives, so it gives rise to a derived functor $R\,\mathrm{Hom} : D^+(\mathfrak{Ab}) \to D^+(\mathfrak{Ab})$. The corresponding classical derived functors $R^i\,\mathrm{Hom}(C, -)$ are usually denoted $\mathrm{Ext}^i(C, -)$.

There is a parallel theory for right-exact functors, giving rise to left derived functors $LF : D^-(\mathcal{A}) \to D^-(\mathcal{B})$. The class of *projectives* takes the place of the class of injectives in this setting. Unfortunately, the categories of greatest interest to us—categories of sheaves on topological spaces—do not, in general, have enough projectives.

Example 13. The category $\mathfrak{Ab}$ does have enough projectives, so for a fixed abelian group C, the right-exact functor $C \otimes -$ gives rise to a derived functor $C \otimes^L -$. The classical derived functors known as "Tor" are defined by $\mathrm{Tor}_i(C, D) = H^{-i}(C \otimes^L D)$. (Note the negative exponent: this comes from the fact that Tor is usually defined in terms of *homological* complexes (in which differentials reduce degree), whereas by convention all of our complexes are *cohomological* (differentials raise degree).)

The examples above, Ext and Tor, are probably the best-known classical examples of derived functors. They also illustrate a shortcoming of the theory we have developed so far: the functors $R\,\mathrm{Hom}(-, -)$ and $- \otimes^L -$ are asymmetric in that they take an object of $\mathfrak{Ab}$ in the first variable, but a complex from $D(\mathfrak{Ab})$ in the second. It would be more natural to allow complexes in both variables.

The correct way to handle this problem is to develop a full theory of derived *bifunctors* (functors of two variables); such a theory should incorporate the idea (familiar from the classical theory of Ext and Tor) that one should be able to compute a derived bifunctor by taking an appropriate resolution of either variable. Such a theory is indeed developed in the book of Kashiwara–Schapira, but we will not develop it here. Instead, we will simply show how to repair the definitions of $R\,\mathrm{Hom}$ and $\otimes^L$ to allow complexes in both variables.

Recall that in general, a left-exact functor gives rise to a sequence of four functors:

$$\begin{aligned} F &: \mathcal{A} \to \mathcal{B} \\ F &: C(\mathcal{A}) \to C(\mathcal{B}) \\ F &: K(\mathcal{A}) \to K(\mathcal{B}) \\ RF &: D^+(\mathcal{A}) \to D^+(\mathcal{B}) \end{aligned}$$

To repair $R\operatorname{Hom}$ and $\otimes^L$, we will interfere with these sequences at the second step: instead of simply considering the functors

$$\operatorname{Hom}(C,-) : C(\mathfrak{Ab}) \to C(\mathfrak{Ab}) \qquad \text{and} \qquad C \otimes - : C(\mathfrak{Ab}) \to C(\mathfrak{Ab})$$

that are induced by functors $\mathfrak{Ab} \to \mathfrak{Ab}$, we replace them by new functors (called "graded Hom" and "graded tensor product") defined only at the level of complexes and not on $\mathfrak{Ab}$ itself.

Let $C^\bullet$ and $D^\bullet$ be complexes of abelian groups. Their graded Hom, denoted $\underline{\operatorname{Hom}}(C^\bullet, D^\bullet)$, is the complex given by

$$\underline{\operatorname{Hom}}(C^\bullet, D^\bullet)^i = \bigoplus_{k-j=i} \operatorname{Hom}(C^j, D^k).$$

The differential $d : \underline{\operatorname{Hom}}(C^\bullet, D^\bullet)^i \to \underline{\operatorname{Hom}}(C^\bullet, D^\bullet)^{i+1}$ can be thought of as a large collection of maps

$$\operatorname{Hom}(C^j, D^k) \to \operatorname{Hom}(C^m, C^n),$$

where $k - j = i$ and $n - m = i + 1$. These small portions of d are defined as follows:

$$(d)_{jk,mn} : \operatorname{Hom}(C^j, D^k) \to \operatorname{Hom}(C^m, D^n)$$

$$(d)_{jk,mn} = \begin{cases} f \mapsto d_D^k \circ f & \text{if } m = j \text{ and } n = k+1, \\ f \mapsto (-1)^j f \circ d_C^{j-1} & \text{if } m = j-1 \text{ and } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

It is left as an exercise to verify that this differential makes $\underline{\operatorname{Hom}}(C^\bullet, D^\bullet)$ into a complex (*i.e.*, that $d^2 = 0$).

Now, $R\operatorname{Hom}(C^\bullet, D^\bullet)$ is simply defined to be $\underline{\operatorname{Hom}}(C^\bullet, I^\bullet)$, where $I^\bullet$ is an injective resolution of $D^\bullet$. The proof that this is well-defined is the same as before.

The definition of the graded tensor product is similar: $C^\bullet \underline{\otimes} D^\bullet$ is the complex given by

$$(C^\bullet \underline{\otimes} D^\bullet)^i = \bigoplus_{j+k=i} (C^j \otimes D^k).$$

As above, the differential can be specified by specifying

$$(d)_{jk,mn} : C^j \otimes D^k \to C^m \otimes D^n$$

$$(d)_{jk,mn} = \begin{cases} \mathrm{id}_{C^j} \otimes d_D^k & \text{if } m = j \text{ and } n = k+1, \\ (-1)^j d_C^j \otimes \mathrm{id}_{D^k} & \text{if } m = j+1 \text{ and } n = k, \\ 0 & \text{otherwise.} \end{cases}$$

The verification that $d^2 = 0$ is again left as an exercise. Then $C^\bullet \otimes^L D^\bullet$ is defined to be $C^\bullet \underline{\otimes} P^\bullet$, where $P^\bullet$ is a projective (or flat) resolution of $D^\bullet$.