Derived Functors in Categories of Sheaves

March 8, 2007

In the philosophy expounded by Grothendieck, there are six important operations on sheaves, occuring in three adjoint pairs:

$$(\overset{L}{\otimes}, R\mathcal{H}om), \qquad (f^{-1}, Rf_*), \qquad (Rf_!, f^!).$$

All other operations can be built out of these. (For example, $R\Gamma$ is the same as Ra_*, where $a : X \to \{\text{pt}\}$ is the constant map to a point, and $R\operatorname{Hom} = R\Gamma \circ R\mathcal{H}om$.) In this set of notes, we will review the definitions of all of these except $f^!$, list adapted classes, and collect various composition and adjointness theorems.

Convention. All statements below are correct for sheaves of abelian groups. However, some of them contain conditions that become superfluous or trivial in the category of sheaves of vector spaces. Specifically, all sheaves of vector spaces are flat, so $\otimes$ is already an exact functor. Nevertheless, we will use the usual notation $\otimes^L$ for the functor induced by $\otimes$ on the derived category.

Theorem 1. *The category $\mathfrak{Sh}_X$ of sheaves on X has enough injectives.*

As a consequence, we can form the derived functor of all left-exact functors on sheaves.

Refer to the previous set of notes for the definitions $\underline{\operatorname{Hom}}$ and $\underline{\otimes}$. Analogous definitions can be made for $\underline{\mathcal{H}om}$ and $\underline{\otimes}$ in the category of sheaves on a space X. Then, we define

$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \underline{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \qquad \text{where } \mathcal{I}^\bullet \text{ is an injective resolution of } \mathcal{G}^\bullet,$$

$$\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet = \mathcal{F}^\bullet \underline{\otimes} \mathcal{P}^\bullet \qquad \text{where } \mathcal{P}^\bullet \text{ is a flat resolution of } \mathcal{G}^\bullet.$$

(Because all sheaves of vector spaces are flat, one can always just take $\mathcal{P}^\bullet = \mathcal{G}^\bullet$.)

Given a continuous map $f : X \to Y$, the push-forward functor f_* is left-exact, so we can form its derived functor Rf_*. The pullback functor f^{-1} is exact and so automatically gives rise to a functor f^{-1} on the derived category.

We need to a bit of preparation to define $f_!$.

Definition 2. Let $\mathcal{F}$ be a sheaf on X, let $s \in \mathcal{F}(U)$. The **support** of s is defined to be

$$\operatorname{supp} s = \{x \in U \mid s_x \neq 0\}.$$

This is automatically a closed subset of U.

Definition 3. A continuous map $f : X \to Y$ is **proper** if, for every compact set $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact.

Definition 4. Let $f : X \to Y$ be a continuous map, and let $\mathcal{F}$ be a sheaf on X. The **proper push-forward** of $\mathcal{F}$, denoted $f_!\mathcal{F}$, is the subsheaf of $f_*\mathcal{F}$ defined by

$$(f_!\mathcal{F})(U) = \{s \in f^{-1}(U) \mid f|_{\operatorname{supp} s} : \operatorname{supp} s \to U \text{ is proper}\}.$$

Note that the restriction of a proper map to a closed subset of its domain is always proper. If f is a proper map, then the functors $f_!$ and f_* coincide, because $f|_{\operatorname{supp} s}$ is always proper. In particular, if f is an inclusion of a closed subset, proper push-forward and ordinary push-forward coincide.

If f is an inclusion of an open subset, $f_!$ coincides with the "extension by zero" functor defined earlier.

Definition 5. A sheaf $\mathcal{F}$ on X is **flabby** (or **flasque**) if the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ are surjective for all U in X.

Definition 6. $\mathcal{F}$ is **soft** if for every compact set $K \subset X$, the natural map $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}|_K)$, or, equivalently, the natural map

$$\mathcal{F}(X) \to \varinjlim_{U \supset K} \mathcal{F}(U),$$

is surjective.

Remark 7. Some sources use the term "c-soft" for sheaves satisfying the above definition, and use the term "soft" for a related notion in which "compact" is replaced by "closed." However, on locally compact, second countable Hausdorff spaces, the two notions coincide.

Proposition 8. *Every injective sheaf is flabby. Every flabby sheaf is soft.*

Theorem 9. *The various functors on sheaves are left-exact, right-exact, or exact as shown in the table below. For each functor, the classes of sheaves listed in the third column are adapted.*

Note that this theorem is *not* asserting that the largest class listed for a given functor is in fact the largest adapted class for that functor.

Functor	*Exactness*	*Adapted classes*	*Derived functor*	*Classical derived functors*
Γ	left	injective, flabby	$R\Gamma$	$H^i(X,-)$
f_*	left	injective, flabby	Rf_*	$R^i f_*$
Hom	left	injective	$R\,\mathrm{Hom}$	Ext^i
$\mathcal{H}om$	left	injective	$R\,\mathcal{H}om$	$\mathcal{E}xt^i$
$f_!$	left	injective, flabby, soft	$Rf_!$	$R^i f_!$
f^{-1}	exact	—	f^{-1}	
$\otimes$	right	flat	$\otimes^L$	$\mathcal{T}or_i$

Once again, recall that for sheaves of vector spaces, $\otimes$ is exact, and all sheaves are flat. Also, the classical derived functor $\mathcal{T}or_i$ (as well as the functor Tor_i for abelian groups) is usually defined *homologically*: it is defined in terms of a resolution whose differentials decrease degree. By convention, all our complexes have differentials that raise degree, so the classical $\mathcal{T}or_i$ is obtained from our $\otimes^L$ by:

$$\mathcal{T}or_i(\mathcal{F},\mathcal{G}) = H^{-i}(\mathcal{F} \overset{L}{\otimes} \mathcal{G}).$$

Since the category $\mathfrak{Sh}_X$ has enough injectives, no other adapted classes would have been needed if all we wanted to do was define the derived functors of the various left-exact functors. However, we also want to be able to understand compositions of derived functors. The following fact tells us how to do this.

Proposition 10. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be two left-exact functors. If there is an adapted class $\mathcal{R}$ for F and an adapted class $\mathcal{S}$ for G such that $F(\mathcal{R}) \subset \mathcal{S}$, then $R(G \circ F) = RG \circ RF$.*

Not every left-exact functor takes injective objects to injective objects, so in order to apply the above proposition, we usually need to have some additional adapted classes at hand. This is where flabby and soft sheaves are useful.

Proposition 11. *The functors f_*, $f_!$, and $\mathcal{H}om$ act on injective, flabby, and soft sheaves as shown here:*

$$\begin{array}{lll} f_* \ \text{(injective)} = \text{injective} & \qquad & \mathcal{H}om(\text{anything}, \text{injective}) = \text{flabby} \\ f_* \ \text{(flabby)} = \text{flabby} & & \mathcal{H}om(\text{flat}, \text{injective}) = \text{injective} \\ f_! \ \text{(soft)} = \text{soft} & & \end{array}$$

The following theorem on compositions is now immediate:

Theorem 12. *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then*

$$Rf_* \circ Rg_* = R(f \circ g)_*$$
$$Rf_! \circ Rg_! = R(f \circ g)_!$$

If $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are complexes of sheaves on X, then

$$R\Gamma(R\,\mathcal{H}om(\mathcal{F}^\bullet, G^\bullet)) = R\,\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Finally, we obtain two of the three important adjointness theorems by using the above facts on compositions of derived functors. In both cases, the proof consists of reducing to the corresponding theorems in the nonderived setting.

Theorem 13. *If* $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^-(\mathfrak{Sh}_X)$ *and* $\mathcal{H}^\bullet \in D^+(\mathfrak{Sh}_X)$*, then we have:*

$$\operatorname{Hom}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq \operatorname{Hom}(\mathcal{F}^\bullet, R\,\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet)),$$
$$R\,\mathcal{H}om(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq R\operatorname{Hom}(\mathcal{F}^\bullet, R\,\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet)),$$
$$R\operatorname{Hom}(\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet, \mathcal{H}^\bullet) \simeq R\,\mathcal{H}om(\mathcal{F}^\bullet, R\,\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{H}^\bullet)).$$

Theorem 14. *If* $f : X \to Y$ *is a continuous map,* $\mathcal{F}^\bullet \in D^-(\mathfrak{Sh}_Y)$*, and* $\mathcal{G}^\bullet \in D^+(\mathfrak{Sh}_X)$*, then we have:*

$$\operatorname{Hom}(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \operatorname{Hom}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$
$$R\operatorname{Hom}(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\operatorname{Hom}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$
$$Rf_*R\,\mathcal{H}om(f^{-1}\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\,\mathcal{H}om(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$