Proper Pull-Back and Poincaré–Verdier Duality

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Convention. We now add an assumption to our standing list of assumptions on topological spaces. For each topological space X, we assume there is a number r such that for any exact sequence

$$\mathcal{F}^0 \to \cdots \to \mathcal{F}^r \to 0$$

of sheaves on X, if $\mathcal{F}^0, \ldots, \mathcal{F}^{r-1}$ are all soft, then so is $\mathcal{F}^r$. (This assumption is true for manifolds and for closed subsets of manifolds.)

We also add a hypothesis on sheaves. From now on, all sheaves will be assumed to be sheaves of *finite-dimensional* vector spaces, and $\mathfrak{Sh}_X$ will denote the category of such sheaves.

The main result in this set of notes is the following.

Theorem 1. *Let $f : X \to Y$ be a continuous map. There is a functor $f^! : D^+(\mathfrak{Sh}_Y) \to D^+(\mathfrak{Sh}_X)$, unique up to isomorphism, that is right-adjoint to $Rf_!$:*

$$\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \operatorname{Hom}\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

The proof relies on the following sequence of lemmas.

Lemma 2. *Any sheaf $\mathcal{F}$ on X has a soft resolution of length at most r.*

Lemma 3. *For any sheaf $\mathcal{F}$ on X, there exists a sheaf $\mathcal{P}$ and a surjective morphism $\mathcal{P} \to \mathcal{F}$ where $\mathcal{P}$ is of the form $\mathcal{P} = \prod_i \underline{\mathbb{C}}_{V_i}$. Here $\{V_i\}$ is some collection of open sets, and $\underline{\mathbb{C}}_{V_i}$ is regarded as a sheaf on X by extension by zero.*

(The proof of the preceding lemma is close to a general proof of the existence of flat resolutions in the category of sheaves of abelian groups.)

Lemma 4. *If $\mathcal{K}$ is a flat, soft sheaf on X, then for any sheaf $\mathcal{F}$ on X, $\mathcal{F} \otimes \mathcal{K}$ is soft.*

Corollary 5. *The functor $\mathcal{F} \mapsto f_!(\mathcal{F} \otimes \mathcal{K})$ is exact.*

Now, if $\mathcal{K}$ is a sheaf on X and $\mathcal{G}$ is a sheaf on Y, let $\mathcal{H}(\mathcal{K}, \mathcal{G})$ be the presheaf given by

$$\mathcal{H}(\mathcal{K}, \mathcal{G})(V) = \operatorname{Hom}(f_!(j_!(\mathcal{K}|_V)), \mathcal{G}) \qquad \text{where } j : V \hookrightarrow X \text{ is the inclusion map.}$$

Lemma 6. *If $\mathcal{K}$ is flat and soft, and $\mathcal{G}$ is injective, then*

(1) *$\mathcal{H}(\mathcal{K}, \mathcal{G})$ is a sheaf.*
(2) $\operatorname{Hom}(f_!(\mathcal{F} \otimes \mathcal{K}), \mathcal{G}) \simeq \operatorname{Hom}(\mathcal{F}, \mathcal{H}(\mathcal{K}, \mathcal{G}))$.
(3) *$\mathcal{H}(\mathcal{K}, \mathcal{G})$ is injective.*

The last step is to actually define $f^!$. Choose a resolution $\mathcal{K}^\bullet$ of $\underline{\mathbb{C}}_X$ by flat, soft modules. (If we had been working in the setting of sheaves of abelian groups, the "flat" part of the preceding sentence would require additional work, but for sheaves of vector spaces, it is automatic.) Let $\mathcal{G}^\bullet \in D^+(\mathfrak{Sh}_X)$. Assume that $\mathcal{G}^\bullet$ is a complex of injective sheaves (*i.e.*, replace it by an injective resolution if necessary. Define the complex $f^!\mathcal{G}^\bullet$ by

$$(f^!\mathcal{G}^\bullet)^i = \bigoplus_{k-j=i} \mathcal{H}(\mathcal{K}^j, \mathcal{G}^k),$$

with differentials defined in the same way as for $\underline{\mathcal{H}om}$. Once the theorem is proved, we also know that $f^!\mathcal{G}^\bullet$ is independent of the choice of $\mathcal{K}^\bullet$ by the following lemma.

Lemma 7 (Uniqueness of adjoints). *If (F, G) and (F, H) are both adjoint pairs of functors, then $G \simeq H$.*

As usual, we also have derived and sheaf-theoretic versions of the adjointness theorem. Note that these cannot be proved by reducing to the nonderived case (as was done for the adjoint pairs $(\otimes^L, R\mathcal{H}om)$ and (f^{-1}, Rf_*)), because there is no nonderived version of $f^!$ in general.

Proposition 8. *Let $f : X \to Y$ be a continuous map. For any $\mathcal{F}^\bullet \in D^b(\mathfrak{Sh}_X)$ and $\mathcal{G}^\bullet \in D^+(\mathfrak{Sh}_Y)$, we have:*

$$R\operatorname{Hom}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\operatorname{Hom}\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet),$$
$$R\mathcal{H}om(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq Rf_*R\mathcal{H}om\,\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

Proposition 9. $f^!R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq R\mathcal{H}om(f^{-1}\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$.

Definition 10. Let $a : X \to \{\mathrm{pt}\}$ be the constant map from X to a one-point space. The **dualizing complex** on X is the complex $\omega_X = a^!\underline{\mathbb{C}}$.

Definition 11. Let $\mathcal{F}^\bullet \in D^-(\mathfrak{Sh}_X)$. The **Verdier dual** of $\mathcal{F}^\bullet$ is the complex $\mathbb{D}\mathcal{F}^\bullet = R\mathcal{H}om(\mathcal{F}, \omega_X)$.

Proposition 12. $f^!\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}f^{-1}\mathcal{F}^\bullet$.

Proposition 13. $\mathbb{D}\mathbb{D}\mathcal{F}^\bullet \simeq \mathcal{F}^\bullet$.

(The preceding proposition is not true if one works with sheaves of possibly infinite dimension, essentially because taking the vector space dual twice gives the original vector space only if the original vector space is finite-dimensional.)

Corollary 14. $f^{-1}\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}f^!\mathcal{F}^\bullet$.

Proposition 15. *We have:*

$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq R\mathcal{H}om(\mathcal{G}^\bullet, \mathbb{D}\mathcal{F}^\bullet) \qquad Rf_!\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_*\mathcal{F}^\bullet$$
$$R\mathcal{H}om(\mathcal{F}^\bullet, \mathbb{D}\mathcal{G}^\bullet) \simeq \mathbb{D}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet) \qquad Rf_*\mathbb{D}\mathcal{F}^\bullet \simeq \mathbb{D}Rf_!\mathcal{F}^\bullet$$

The following theorem is an easy consequence of the adjointness of $Rf_!$ and $f^!$ in the special case where $f : X \to \{\mathrm{pt}\}$ is the map to a point.

Theorem 16 (Verdier Duality)**.** $H_c^{-i}(X, \mathcal{F}^\bullet)^* \simeq H^i(X, \mathbb{D}\mathcal{F}^\bullet)$.

Classical Poincaré duality can be recovered from this with the aid of the following fact.

Proposition 17. *Let X be a smooth, oriented n-dimensional manifold. Then $\omega_X \simeq \underline{\mathbb{C}}_X[n]$.*

It follows immediately from this that on a smooth, oriented n-dimensional manifold, we have $\mathbb{D}\underline{\mathbb{C}} \simeq \underline{\mathbb{C}}[n]$. Then the Verdier duality theorem above becomes the following.

Theorem 18 (Poincaré Duality)**.** $H_c^{n-i}(X, \mathbb{C})^* \simeq H^i(X, \mathbb{C})$.

Historically, of course, Poincaré duality came long before Verdier duality, and Verdier duality can be seen as a generalization of Poincaré duality.

Indeed, Verdier duality holds in great generality, whereas Poincaré duality, a much more specific statement, fails for most spaces that are not manifolds. For most spaces, and most complexes of sheaves, the complexes $\mathcal{F}^\bullet$ and $\mathbb{D}\mathcal{F}^\bullet$ that appear in Verdier duality are different. What makes Poincaré duality work for manifolds is the fact that the constant sheaf is close to self-dual. (Its dual is just a shift of itself.)

If we could find self-dual complexes of sheaves on other spaces, then we could achieve a sort of intermediate generalization of Poincaré duality: a duality theorem that is closer in spirit to the original Poincaré duality, but yet that holds on many spaces that are not manifolds.

The search for such self-dual complexes of sheaves is one of the principal motivations for the development of the theory of perverse sheaves.