Triangulated Categories and t-Structures

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Definition 1. A **triangulated category** is an additive category $\mathcal{C}$ equipped with (a) a **shift functor** $[1] : \mathcal{C} \to \mathcal{C}$ and (b) a class of triangles $X \to Y \to Z \to X[1]$, called **distinguished triangles**, satisfying the following axioms:

(-2) The shift functor is an equivalence categories. In particular, it has an inverse, denoted $[-1] : \mathcal{C} \to \mathcal{C}$.
(-1) Every triangle isomorphic to a distinguished triangle is a distinguished triangle.
(0) Every morphism $f : X \to Y$ can be completed to a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.
(1) (Identity)
(2) (Rotation)
(3) (Square Completion)
(4) (Octahedral Property)

(The last four axioms are the properties of distinguished triangles in the derived category that were established in an earlier set of notes.)

Example 2. The derived category of an abelian category, with its usual shift functor and its usual notion of distinguished triangles, is a triangulated category. The same is true of the homotopy category of complexes over an abelian category. Indeed, these are essentially the only examples of triangulated categories that we will see.

Definition 3. Let $\mathcal{C}$ be a triangulated category. A ***t*-structure** on $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}^{\le 0}, \mathcal{C}^{\ge 0})$ satisfying the axioms below. (For any $n \in \mathbb{Z}$, we use the notation $\mathcal{C}^{\le n} = \mathcal{C}^{\le 0}[-n]$, $\mathcal{C}^{\ge n} = \mathcal{C}^{\ge 0}[-n]$.)

(1) $\mathcal{C}^{\le 0} \subset \mathcal{C}^{\le 1}$ and $\mathcal{C}^{\ge 0} \supset \mathcal{C}^{\ge 1}$.
(2) $\bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\le n} = \bigcap_{n\in\mathbb{Z}} \mathcal{C}^{\ge n} = \{0\}$.
(3) If $A \in \mathcal{C}^{\le 0}$ and $B \in \mathcal{C}^{\ge 1}$, then $\mathrm{Hom}(A, B) = 0$.
(4) For any object X in $\mathcal{C}$, there is a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{C}^{\le 0}$ and $B \in \mathcal{C}^{\ge 1}$.

Although the last axiom does not say anything about uniqueness of the distinguished triangle, it turns out to be unique as a consequence of the other axioms. Specifically, we have:

Proposition 4. *The distinguished triangle in Axiom* (4) *above is unique up to isomorphism. Indeed, there are functors* ${}^t\tau_{\le 0} : \mathcal{C} \to \mathcal{C}^{\le 0}$ *and* ${}^t\tau_{\ge 1} : \mathcal{C} \to \mathcal{C}^{\ge 1}$ *such that for any object* X *of* $\mathcal{C}$,

$$ {}^t\tau_{\le 0}X \to X \to {}^t\tau_{\ge 1}X \to ({}^t\tau_{\le 0}X)[1] $$

is that distinguished triangle.

Proposition 5. *Let* $\iota_{\le 0} : \mathcal{C}^{\le 0} \to \mathcal{C}$ *be the inclusion functor. Then* $(\iota_{\le 0}, {}^t\tau_{\le 0})$ *is an adjoint pair.*
Similarly, let $\iota_{\ge 1} : \mathcal{C}^{\ge 1} \to \mathcal{C}$) *be the inclusion functor. Then* $({}^t\tau_{\ge 0}, \iota_{\ge 0})$ *is an adjoint pair.*

When doing calculations in a triangulated category with a t-structure, the preceding proposition typically comes up in the following way: if $A \in \mathcal{C}^{\le 0}$ and $f : A \to X$ is any morphism in $\mathcal{C}$, then f factors through ${}^t\tau_{\le 0}X$. That is, there is a unique morphism f' making the following diagram commute:

$$\begin{tikzcd} A \arrow[dr, dashed, "f'"'] \arrow[drr, bend left, "f"] & & \\ & {}^t\tau_{\le 0}X \arrow[r] & X \end{tikzcd}$$

Similarly, if $B \in \mathcal{C}^{\ge 1}$ and $g : X \to B$ is any morphism, then g factors through ${}^t\tau_{\ge 1}X$:

$$\begin{tikzcd} & & B \\ X \arrow[r] \arrow[urr, bend left, "g"] & {}^t\tau_{\ge 1}X \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

Of course, although the last two propositions were stated in terms of "≤ 0" and "≥ 1," there are corresponding statements with "$\leq n$" and "$\geq n$" for any $n \in \mathbb{Z}$, obtained by shifting. In particular, there are truncation functors ${}^t\tau_{\leq n}$ and ${}^t\tau_{\geq n}$ for all n, and there are distinguished triangles

$$ {}^t\tau_{\leq n}X \to X \to {}^t\tau_{\geq n+1}X \to ({}^t\tau_{\leq n}X)[1] $$

for all n. The relationship between truncation and shifting is given by:

$$ {}^t\tau_{\leq n}X = ({}^t\tau_{\leq 0}(X[n]))[-n]. $$

Proposition 6. *Suppose $n \leq m$. Then we have*

$$ \begin{aligned} &{}^t\tau_{\leq n}{}^t\tau_{\leq m} = {}^t\tau_{\leq m}{}^t\tau_{\leq n} = {}^t\tau_{\leq n}, && {}^t\tau_{\geq n}{}^t\tau_{\leq m} = {}^t\tau_{\leq m}{}^t\tau_{\geq n} \\ &{}^t\tau_{\geq n}{}^t\tau_{\geq m} = {}^t\tau_{\geq m}{}^t\tau_{\geq n} = {}^t\tau_{\geq m}, && {}^t\tau_{\leq n}{}^t\tau_{\geq m} = {}^t\tau_{\geq m}{}^t\tau_{\leq n} = 0 \quad \text{if } n < m. \end{aligned} $$

In particular, all truncation functors commute with each other.

Definition 7. Let $\mathcal{C}$ be a triangulated category with a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$. The category $\mathcal{T} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ is called the **heart** (or **core**) of the t-structure.
The functor ${}^tH^0 : \mathcal{C} \to \mathcal{T}$ defined by ${}^tH^0 = {}^t\tau_{\leq 0}{}^t\tau_{\geq 0} = {}^t\tau_{\geq 0}{}^t\tau_{\leq 0}$ is called **(zeroth) *t*-cohomology**. Moreover, for any $i \in \mathbb{Z}$, the functor ${}^tH^i$ defined by ${}^tH^i(X) = {}^tH^0(X[i])$ (or, equivalently, ${}^tH^i = {}^t\tau_{\leq i}{}^t\tau_{\geq i} = {}^t\tau_{\geq i}{}^t\tau_{\leq i}$ is called the i**th *t*-cohomology**.

The following theorem is the reason we want to introduce the notion of t-structures.

Theorem 8. *Let $\mathcal{C}$ be a triangulated category with a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$, and let $\mathcal{T}$ be its heart. $\mathcal{T}$ is an abelian category. Moreover, the functor ${}^tH^0 : \mathcal{C} \to \mathcal{T}$ with the following properties:*

1. *For any object $A \in \mathcal{T}$, ${}^tH^0(A) \simeq A$.*
2. *${}^tH^0$ takes distinguished triangles in $\mathcal{C}$ to long exact sequences in $\mathcal{T}$.*
3. *A morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism if and only if the morphisms ${}^tH^i(f)$ are isomorphisms in $\mathcal{T}$ for all $i \in \mathbb{Z}$.*
4. *We have*

$$ \begin{aligned} \mathcal{C}^{\leq 0} &= \{X \in \mathcal{C} \mid {}^tH^i(X) = 0 \text{ for all } i > 0\}, \\ \mathcal{C}^{\geq 0} &= \{X \in \mathcal{C} \mid {}^tH^i(X) = 0 \text{ for all } i < 0\}. \end{aligned} $$