Gluing t-Structures; Perverse Sheaves

March 29, 2007

Notation 1. Henceforth, the derived category of sheaves on a topological space X will be denoted $D(X)$ instead of $D(\mathfrak{Sh}_X)$. Its various bounded versions will similarly be denoted $D^+(X)$, $D^-(X)$, and $D^b(X)$. The standard t-structure on $D^b(X)$ will be denoted $({}^{\mathrm{std}}D^b(X)^{\leq 0}, {}^{\mathrm{std}}D^b(X)^{\geq 0})$.

Let X be a topological space; let $U \subset X$ be an open set, and $Z = X \smallsetminus U$ its complement. Let $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ be the inclusion maps. Consider the bounded derived categories $D^b(U)$, $D^b(Z)$, and $D^b(X)$. We have the following six functors:

$$\begin{aligned} Rj_* &: D^b(U) \to D^b(X) & \qquad Ri_* &: D^b(Z) \to D^b(X) \\ Rj_! &: D^b(U) \to D^b(X) & i^! &: D^b(X) \to D^b(Z) \\ j^{-1} &: D^b(X) \to D^b(U) & i^{-1} &: D^b(X) \to D^b(Z) \end{aligned}$$

(Note that $Ri_! = Ri_*$, since a closed inclusion is a proper map, and $j^! = j^{-1}$ by Exercise 4 on Problem Set 7.)

The following facts are useful to keep in mind: for any sheaf $\mathcal{F}$, there is a distinguished triangle

$$Rj_!j^{-1}\mathcal{F} \to \mathcal{F} \to Ri_*i^{-1}\mathcal{F} \to (Rj_!j^{-1}\mathcal{F})[1].$$

Using the Verdier duality functor $\mathbb{D}$, we also obtain a distinguished triangle

$$Ri_*i^!\mathcal{F} \to \mathcal{F} \to Rj_*j^{-1}\mathcal{F} \to (Ri_*i^!\mathcal{F})[1].$$

(Take the first distinguished triangle for the sheaf $\mathbb{D}\mathcal{F}$, then apply $\mathbb{D}$ to the whole triangle, keeping in mind the interaction of $\mathbb{D}$ with the various push-forward and pull-back functors.)

Also, it is obvious that

$$i^{-1}Rj_!\mathcal{F} = 0$$

for any sheaf $\mathcal{F}$ on U, by the definition of "extension by zero." Another Verdier duality argument yields that

$$i^!Rj_*\mathcal{F} = 0$$

as well.

Now, let D_U be a full subcategory of $D^b(U)$, D_Z a full subcategory of $D^b(Z)$, and D a full subcategory of $D^b(X)$, such that the following conditions hold:

- D_U, D_Z, and D are triangulated categories. (In particular, they are stable under the shift operation and under formation of distinguished triangles.)
- The six functors above still "work." (That is, $Rj_*|_{D_U}$ should take values in D, *etc.*)

As an example, one can certainly take $D_U = D^b(U)$, $D_Z = D^b(Z)$, and $D = D^b(X)$, but later on we will make a different choice for these categories.

Theorem 2 (Gluing t-structures)**.** *Given t-structures $(D_U^{\leq 0}, D_U^{\geq 0})$ and $(D_Z^{\leq 0}, D_Z^{\geq 0})$ on D_U and D_Z, respectively, there is a t-structure on D defined by*

$$\begin{aligned} D^{\leq 0} &= \{\mathcal{F} \mid j^{-1}\mathcal{F} \in D_U^{\leq 0} \text{ and } i^{-1}\mathcal{F} \in D_Z^{\leq 0}\} \\ D^{\geq 0} &= \{\mathcal{F} \mid j^{-1}\mathcal{F} \in D_U^{\geq 0} \text{ and } i^{!}\mathcal{F} \in D_Z^{\geq 0}\} \end{aligned}$$

The main idea in the definition of perverse sheaves is that using the above theorem, we want to define a t-structure on the derived category of sheaves on a stratified space by gluing together various shifts of the standard t-structure on each stratum.

Definition 3. A **stratification** of a topological space X is a finite set $\mathcal{S}$ of subspaces (called **strata**) of X such that:

- X is the disjoint union of all the strata.

- Each stratum $S \in \mathcal{S}$ is a manifold.
- The closure of a stratum $\overline{S}$ is a union of strata.

We also call X a **stratified space**.

(Later, we will impose an additional condition on our stratifications.)

If X is a stratified space, then its set of strata $\mathcal{S}$ carries a natural partial order: we say that $S \leq T$ if and only if $S \subset \overline{T}$. Since $\mathcal{S}$ is finite, there must obviously be strata that are minimal with respect to this partial order, and others that are maximal. Evidently, a stratum is minimal if and only if it is closed. A stratum is maximal if and only if it is open.

Definition 4. An ordinary sheaf $\mathcal{F}$ on a stratified space X is **constructible** with respect to the stratification $\mathcal{S}$ if for all $S \in \mathcal{S}$, the restriction $\mathcal{F}|_S$ is locally constant.

A complex of sheaves $\mathcal{F}$ is said to be **constructible** if all of its cohomology sheaves $H^i(\mathcal{F})$ are constructible in the above sense.

The full subcategory of $D^b(X)$ consisting of constructible sheaves is denoted $D^b_c(X)$.

The category $D^b_c(X)$ is often casually referred to as "the derived category of constructible sheaves," although this is somewhat of a misnomer: it is a subcategory of the derived category of the category of ordinary sheaves, and it is not equivalent to the derived category of the category of ordinary constructible sheaves.

Definition 5. Let X be a stratified space, with set of strata $\mathcal{S}$. A **perversity function** is simply a function $p : \mathcal{S} \to \mathbb{Z}$.

For each stratum $S \in \mathcal{S}$, let $i_S : S \hookrightarrow X$ be the inclusion map. The **perverse *t*-structure** on $D = D^b(X)$ with respect to the perversity p is given by

$$ {}^pD^{\leq 0} = \{\mathcal{F} \mid i_S^{-1}\mathcal{F} \in {}^{\mathrm{std}}D^b(S)^{\leq p(S)} \text{ for all } S \in \mathcal{S}\} $$
$$ {}^pD^{\geq 0} = \{\mathcal{F} \mid i_S^{!}\mathcal{F} \in {}^{\mathrm{std}}D^b(S)^{\geq p(S)} \text{ for all } S \in \mathcal{S}\} $$

(This is a slightly nonstandard definition; the issue will be clarified later.) The associated truncation and t-cohomology functors are denoted ${}^p\tau_{\leq 0}$, ${}^p\tau_{\geq 0}$, and ${}^pH^0$. A **perverse sheaf** on X with respect to p is a constructible sheaf in the heart of this t-structure.

Of course, we must show that this actually is a t-structure. That is quite easy; we just use the gluing theorem and induction on the number of strata.

Theorem 6. *The perverse t-structure on $D^b(X)$ is a t-structure.*

Remark 7. By far, the most common setting for perverse sheaves is one in which all strata are even-dimensional, and the perversity function used is $p(S) = -\frac{1}{2}\dim S$. The advantages of this will become clear when we discuss Verdier duality for perverse sheaves.

In any situation in which the perversity function is not specified, it should be assumed that this is the perversity function being used.

One reason the theory of perverse sheaves is so useful is that there is a new kind of extension functor with very good properties.

Theorem 8. *In the context of Theorem 2, let $\mathcal{T}$, $\mathcal{T}_U$, and $\mathcal{T}_Z$ be the hearts of the t-structures on D, D_U, and D_Z, respectively. Let $\mathcal{F}$ be an object of $\mathcal{T}_U$. There is a unique object $\mathcal{G}$ of $\mathcal{T}$, up to isomorphism, such that*

$$ j^{-1}\mathcal{G} \simeq \mathcal{F}, \qquad i^{-1}\mathcal{G} \in D_Z^{\leq -1}, \qquad i^!\mathcal{G} \in D_Z^{\geq 1}. $$

*This object is denoted by $\mathcal{G} = j_{!*}\mathcal{F}$ and is called the **middle extension** or **Goresky–MacPherson extension** of $\mathcal{F}$.*

In addition, let us define two t-structures $(D^{\pm,\leq 0}, D^{\pm,\geq 0})$ on D obtained by gluing $(D_U^{\leq 0}, D_U^{\geq 0})$ and one of $(D_Z^{\leq \pm 1}, D_Z^{\geq \pm 1})$, and let $\tau^{\pm}_{\leq 0}$, $\tau^{\pm}_{\geq 0}$ be the corresponding truncation functors. Then the middle extension can be characterized as

$$ j_{!*}\mathcal{F} \simeq \tau^{-}_{\leq 0} Rj_*\mathcal{F} \simeq \tau^{+}_{\geq 0} Rj_!\mathcal{F}. $$

Definition 9. Let X be a stratified space with set of strata $\mathcal{S}$ and perversity function $p : \mathcal{S} \to \mathbb{Z}$. Let S be a stratum of X, and let $\mathcal{E}$ be a local system (locally constant ordinary sheaf) on S. Let $j_S : S \hookrightarrow \overline{S}$ be the inclusion of S into its closure, and let $i_{\overline{S}} : \overline{S} \hookrightarrow X$ be the inclusion of $\overline{S}$ into X. $\mathcal{E}[p(S)]$ is a perverse sheaf on S, so $j_{S!*}(\mathcal{E}[p(S)])$ is a perverse sheaf on $\overline{S}$, and the sheaf

$$\mathrm{IC}^p(\overline{S}, \mathcal{E}) = i_{\overline{S}*} j_{S!*}(\mathcal{E}[p(S)])$$

is a perverse sheaf on X. (If the perversity function to be used is unambiguous, the "p" may be omitted from the notation.) A perverse sheaf obtained in this way is called an **intersection cohomology complex**. In the case $\mathcal{E} = \underline{\mathbb{C}}_S$, $\mathrm{IC}(\overline{S}, \underline{\mathbb{C}}_S)$ is also denoted $\mathrm{IC}(\overline{S})$ or $\mathrm{IC}_{\overline{S}}$.

But wait! It is clear that intersection cohomology complexes are objects in the heart of the perverse t-structure, but are they constructible?

This question will be resolved in the next set of notes. At the same time, we will deal with the three issues that were put off until "later" in this set of notes.