Problem Set 1

1. Prove the following statements (I did some of these in class). $\mathcal{F}$ and $\mathcal{G}$ are sheaves unless otherwise specified.

   (a) Let $\{V_i\}$ be an open cover of U, and let $s, t \in \mathcal{F}(U)$. If $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$. (In particular, if $s|_{V_i} = 0$ for all i, then $s = 0$.)

   (b) A section is determined by its germs. That is, if $s, t \in \mathcal{F}(U)$ are sections such that $s_x = t_x$ for all $x \in U$, then $s = t$.

   (c) Suppose $\mathcal{F}$ is presheaf, $\mathcal{G}$ is a sheaf, and $\mathcal{F} \subset \mathcal{G}$. Define a subsheaf $\mathcal{F}' \subset \mathcal{G}$ by
   $$\mathcal{F}'(U) = \{s \in \mathcal{G}(U) \mid \text{there is a covering } \{V_i\} \text{ of } U \text{ such that } s|_{V_i} \in \mathcal{F}(V_i) \text{ for all } i\}.$$
   Then $\mathcal{F}^+ \simeq \mathcal{F}'$. In particular, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}^+ \simeq \mathcal{F}$.

   (d) Let $\mathcal{F}$ be a presheaf. The stalks of $\mathcal{F}^+$ are isomorphic to those of $\mathcal{F}$.

   (e) Given a morphism $f : \mathcal{F} \to \mathcal{G}$, where $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf, there is a unique morphism $f^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $f = f^+ \circ \iota$, where ι is the canonical morphism $\mathcal{F} \to \mathcal{F}^+$.

   (f) A morphism $f : \mathcal{F} \to \mathcal{G}$ is injective if and only if $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set U. (*Warning*: the corresponding statement for surjective morphisms is not true.)

   (g) A morphism $f : \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective, an isomorphism) if and only if $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective, an isomorphism) for all $x \in X$.

2. Let $X = \mathbb{C}$. Let $\underline{\mathbb{Z}}$ be the constant sheaf on X with stalk $\mathbb{Z}$. Let $\mathcal{F}$ and $\mathcal{G}$ be the sheaves on X defined as follows:
   $$\mathcal{F}(U) = \{\text{continuous functions } U \to \mathbb{C}\} \qquad \text{and} \qquad \mathcal{G}(U) = \{\text{continuous functions } U \to \mathbb{C} \smallsetminus \{0\}\}.$$
   (Note that $\mathcal{G}(U)$ must be regarded as an abelian group with respect to multiplication, not addition.) Next, we define morphisms $f : \underline{\mathbb{Z}} \to \mathcal{F}$ and $g : \mathcal{F} \to \mathcal{G}$ as follows:
   $$f_U(s) = 2\pi i s \qquad \text{where } s \in \underline{\mathbb{Z}}(U) \text{ is a locally constant function } U \to \mathbb{Z}$$
   $$g_U(s) = e^s \qquad \text{where } s \in \mathcal{F}(U) \text{ is a continuous function } U \to \mathbb{C}$$
   Show that
   $$0 \to \underline{\mathbb{Z}} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G} \to 0$$
   is a short exact sequence of sheaves. Also, show that the presheaf-image of $g : \mathcal{F} \to \mathcal{G}$ is not a sheaf.

3. Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be an exact sequence of sheaves on X, and let $U \subset X$ be an open set. Show that
   $$0 \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{H})$$
   is exact. In other words, $\Gamma(U, \cdot)$ is a left-exact functor from the category of sheaves on X to the category of abelian groups. Show by example that $\Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{H})$ need not be surjective even if $\mathcal{G} \to \mathcal{H}$ is. (*Hint*: Consider the morphism $g : \mathcal{F} \to \mathcal{G}$ of the previous problem.)

4. Let $\mathcal{H}$ be a subsheaf of $\mathcal{F}$. How should one define the quotient sheaf $\mathcal{F}/\mathcal{H}$? (There is an obvious first guess you can make; is that object actually a sheaf, or only a presheaf that needs to be sheafified?) Prove the "first isomorphism theorem": given a morphism $f : \mathcal{F} \to \mathcal{G}$, we have $\mathcal{F}/\ker f \simeq \operatorname{im} f$.