1. A functor $F : C_1 \to C_2$ between two triangulated categories with $t$-structures is said to be $t$-exact if $F(C_1^0) \subset C_2^0$ and $F(C_1^{<0}) \subset C_2^{<0}$. Let $D_U$, $D_Z$, and $D$ be categories of sheaves as in the theorem on gluing of $t$-structures. Show that the $t$-structure on $D$ described in that theorem is the unique $t$-structure on $D$ such that $Ri_* : D_Z \to D$ and $j^{-1} : D \to D_U$ are $t$-exact functors.

2. Suppose we give $D_U$ and $D_Z$ the standard $t$-structure. Show that the $t$-structure on $D$ described by the gluing theorem is the standard $t$-structure. Also, show that the middle-extension functor $j_{!*}$ coincides in this case with the (non-derived) extension-by-zero functor $j_!$.

3. Suppose $D_U$ has the standard $t$-structure. By shifting the standard $t$-structure on $D_Z$ and then gluing, can you obtain a $t$-structure on $D$ for which the middle-extension functor coincides with the non-derived push-forward $j_*$? How about $Rj_*$? $Rj^!$? (Hint: The answers for $j_*$ and $Rj^!$ are "yes." For $Rj_{!*}$, it depends on properties of the topological space $U$. You should find a condition on $U$ under which the answer is "yes.")

4. Let $X = \mathbb{C}$, and let $\mathcal{O}$ be the ordinary sheaf of holomorphic functions on $X$ (in terms of the variable $z$). Now, let $\mathcal{F}$ be the complex of sheaves on $X$ given by

$$\mathcal{F}^{-1} = \mathcal{F}^0 = \mathcal{O}, \quad \mathcal{F}^j = 0 \text{ if } j \neq -1, 0,$$

with the differential $d : \mathcal{F}^{-1} \to \mathcal{F}^0$ given by $d^{-1}(s) = z \frac{d}{dz} s$ for $s \in \mathcal{O}(U)$. Show that

$$H^{-1}(\mathcal{F}) \simeq \mathbb{C}_X \quad \text{and} \quad H^0(\mathcal{F}) \simeq i_* \mathbb{C}_{\{0\}}$$

where $i : \{0\} \hookrightarrow X$ is the inclusion map. (That is, $H^0(\mathcal{F})$ is the skyscraper sheaf at the point 0 with stalk $\mathbb{C}$.) Then, show that $\mathcal{F}$ is a perverse sheaf with respect to the stratification $X = (\mathbb{C} \setminus \{0\}) \coprod \{0\}$. (Hint: To show that $i^! \mathcal{F} \in D^+_Z$, use adjointness theorems to reduce to the problem of showing that $\text{Hom}(i_* \mathcal{G}, \mathbb{C}_X) = 0$ for any ordinary sheaf $\mathcal{G}$ on $Z = \{0\}$.)

5. Find a Jordan–Hölder series for the perverse sheaf $\mathcal{F}$ of the previous exercise. That is, find a sequence of sub-perverse-sheaves $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ such that each quotient $\mathcal{F}_k/\mathcal{F}_{k-1}$ is a simple perverse sheaf. (Hint: Form the distinguished triangle

$$\tau_{< -1} \mathcal{F} \to \mathcal{F} \to \tau_{\geq 0} \mathcal{F} \to (\tau_{< -1} \mathcal{F})[1]$$

with respect to the standard $t$-structure on $D$, and then take the associated long exact sequence of perverse cohomology sheaves.)

6. We have defined $D^b_c(X)$ to be the subcategory of $D^b(X)$ consisting of complexes of sheaves all of whose cohomology sheaves are constructible. Show by example that this is not the same as the derived category of the category of constructible sheaves. (Hint: Consider the sheaf $\mathcal{F}$ of the previous exercise. Show that this sheaf is not quasi-isomorphic to any complex of constructible sheaves.)