Problem Set 10

March 29, 2007

1. A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between two triangulated categories with t-structures is said to be ***t*-exact** if $F(\mathcal{C}_1^{\le 0}) \subset \mathcal{C}_2^{\le 0}$ and $F(\mathcal{C}_1^{\ge 0}) \subset \mathcal{C}_2^{\ge 0}$. Let D_U, D_Z, and D be categories of sheaves as in the theorem on gluing of t-structures. Show that the t-structure on D described in that theorem is the *unique* t-structure on D such that $Ri_* : D_Z \to D$ and $j^{-1} : D \to D_U$ are t-exact functors.

2. Suppose we give D_U and D_Z the standard t-structure. Show that the t-structure on D described by the gluing theorem is the standard t-structure. Also, show that the middle-extension functor $j_{!*}$ coincides in this case with the (non-derived) extension-by-zero functor $j_!$.

3. Suppose D_U has the standard t-structure. By shifting the standard t-structure on D_Z and then gluing, can you obtain a t-structure on D for which the middle-extension functor coincides with the non-derived push-forward j_*? How about Rj_*? $Rj_!$? (*Hint*: The answers for j_* and $Rj_!$ are "yes." For Rj_*, it depends on properties of the topological space U. You should find a condition on U under which the answer is "yes.")

4. Let $X = \mathbb{C}$, and let $\mathcal{O}$ be the ordinary sheaf of holomorphic functions on X (in terms of the variable z). Now, let $\mathcal{F}$ be the complex of sheaves on X given by
$$\mathcal{F}^{-1} = \mathcal{F}^0 = \mathcal{O}, \qquad \mathcal{F}^j = 0 \quad \text{if } j \neq -1, 0,$$
with the differential $d^{-1} : \mathcal{F}^{-1} \to \mathcal{F}^0$ given by $d_U^{-1}(s) = z\frac{d}{dz}s$ for $s \in \mathcal{O}(U)$. Show that
$$H^{-1}(\mathcal{F}) \simeq \underline{\mathbb{C}}_X \qquad \text{and} \qquad H^0(\mathcal{F}) \simeq i_*\underline{\mathbb{C}}_{\{0\}}$$
where $i : \{0\} \hookrightarrow X$ is the inclusion map. (That is, $H^0(\mathcal{F})$ is the skyscraper sheaf at the point 0 with stalk $\mathbb{C}$.) Then, show that $\mathcal{F}$ is a perverse sheaf with respect to the stratification $X = (\mathbb{C} \smallsetminus \{0\}) \coprod \{0\}$. (*Hint*: To show that $i^!\mathcal{F} \in D_Z^{\ge 0}$, use adjointness theorems to reduce to the problem of showing that $\mathrm{Hom}(i_*\mathcal{G}, \underline{\mathbb{C}}_X) = 0$ for any ordinary sheaf $\mathcal{G}$ on $Z = \{0\}$.)

5. Find a Jordan–Hölder series for the perverse sheaf $\mathcal{F}$ of the previous exercise. That is, find a sequence of sub-perverse-sheaves $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ such that each quotient $\mathcal{F}_k/\mathcal{F}_{k-1}$ is a simple perverse sheaf. (*Hint*: Form the distinguished triangle
$$\tau_{\le -1}\mathcal{F} \to \mathcal{F} \to \tau_{\ge 0}\mathcal{F} \to (\tau_{\le -1}\mathcal{F})[1]$$
with respect to the *standard* t-structure on D, and then take the associated long exact sequence of perverse cohomology sheaves.)

6. We have defined $D_c^b(X)$ to be the subcategory of $D^b(X)$ consisting of complexes of sheaves all of whose cohomology sheaves are constructible. Show by example that this is not the same as the derived category of the category of constructible sheaves. (*Hint*: Consider the sheaf $\mathcal{F}$ of the previous exercise. Show that this sheaf is not quasi-isomorphic to any complex of constructible sheaves.)